

Generalised Gelfand Spectra of Nonabelian Unital C^* -Algebras I: Categorical Aspects, Automorphisms and Jordan Structure

Andreas Döring

University of Oxford

andreas.doering@cs.ox.ac.uk

11. December 2012

Abstract

To each unital C^* -algebra \mathcal{A} we associate a presheaf $\underline{\Sigma}^{\mathcal{A}}$, called the *spectral presheaf of \mathcal{A}* , which can be regarded as a generalised Gelfand spectrum. We present some categorical aspects and clarify how much algebraic information about a C^* -algebra is contained in its spectral presheaf. A nonabelian unital C^* -algebra \mathcal{A} that is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$ is determined by its spectral presheaf up to quasi-Jordan isomorphisms. For a particular class of unital C^* -algebras, including all von Neumann algebras with no type I_2 -summand, the spectral presheaf determines the Jordan structure up to isomorphisms. In a companion paper [17], we consider time evolution of quantum systems in the Heisenberg picture and the Schrödinger picture in a formulation based on flows on the spectral presheaf.

Mathematical Subject Classification: Primary 46L05, 18F20; Secondary 81R15, 81R60.

Keywords: C^* -algebra, von Neumann algebra, spectrum, duality, automorphism, presheaf, topos

1 Introduction

We associate a presheaf $\underline{\Sigma}^{\mathcal{A}}$, called the spectral presheaf, to each unital C^* -algebra \mathcal{A} in section 2. This object was first defined in the topos approach to quantum theory [35, 36, 21, 22, 23, 24, 12, 13, 25, 14, 15, 26, 18, 16], see also [32, 33, 34, 42], and is interpreted physically as a generalised state space of a quantum system. Mathematically, the spectral presheaf can be regarded as a generalised Gelfand spectrum of a nonabelian C^* -algebra.

It is shown in section 3 that every unital $*$ -homomorphism between unital C^* -algebras gives rise to a morphism between their presheaves in the opposite direction.

We give some categorical background to this construction in section 4 and introduce a notion of ‘local duality’, which connects the present article to work by Heunen, Landsman, Spitters and Wolters. We show that there is a contravariant functor from the category of unital C^* -algebras and unital $*$ -homomorphisms to a suitable category of presheaves in which the spectral presheaves of the algebras lie. This presheaf category extends to a category of topoi. The construction presented here generalises the contravariant Gelfand-Naimark correspondence between unital abelian C^* -algebras and compact Hausdorff spaces to nonabelian C^* -algebras and suitable generalised spaces (although we will defer the discussion of topologies on the spectral presheaf and of continuity to future work). Analogous results hold for von Neumann algebras.

In section 5, we focus on the automorphisms of the spectral presheaf and determine how much algebraic information about a nonabelian C^* -algebra is contained in its spectral presheaf. For a von Neumann algebra \mathcal{N} without type I_2 -summand, we show that the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$ determines exactly the Jordan structure of \mathcal{N} . Using recent results by Hamhalter, we show that for a unital C^* -algebra \mathcal{A} , the spectral presheaf determines \mathcal{A} up to quasi-Jordan isomorphisms and, in some cases, also up to Jordan isomorphisms.

In a companion paper [17], we consider an application of the results in the present article in physics. The spectral presheaf is considered as a state space for a quantum system, and flows on the spectral presheaf and on associated structures are defined. These flows allow describing the Schrödinger picture and the Heisenberg picture of time evolution of the quantum system in a new, more geometric manner than in standard quantum theory.

This work connects aspects of the theory of C^* -algebras and von Neumann algebras with aspects of category and topos theory. Standard references on operator algebras are e.g. [38, 9], and [39, 37] on topos theory.

2 Unital C^* -algebras and their spectral presheaves

Let \mathbf{uC}^* be the category of unital C^* -algebras with unital $*$ -homomorphisms as arrows. The category \mathbf{ucC}^* of abelian unital C^* -subalgebras with unital $*$ -homomorphisms as arrows is a full and faithful subcategory of \mathbf{uC}^* .

Let $\mathcal{A} \in \text{Ob}(\mathbf{uC}^*)$ be a unital C^* -algebra, and let $\mathcal{C}(\mathcal{A})$ be the set of unital abelian C^* -subalgebras of \mathcal{A} that share the unit element 1 with \mathcal{A} . By convention, we exclude the trivial subalgebra $C_0 := \mathbb{C}1$ from $\mathcal{C}(\mathcal{A})$. The set $\mathcal{C}(\mathcal{A})$ can be partially ordered under inclusion of smaller abelian subalgebras into larger ones. The poset $(\mathcal{C}(\mathcal{A}), \subseteq)$ is called the *context category* of \mathcal{A} . Usually, we will just write $\mathcal{C}(\mathcal{A})$ instead of $(\mathcal{C}(\mathcal{A}), \subseteq)$.

If we explicitly want to include the trivial algebra, we will write $\mathcal{C}(\mathcal{A})_0$ for $\mathcal{C}(\mathcal{A}) \cup \{\mathbb{C}1\}$, which is a poset under inclusion in which $\mathbb{C}1$ is the bottom element.

Remark 1 *Physically, the contexts, i.e., the elements of $C \in \mathcal{C}(\mathcal{A})$, are interpreted as ‘classical perspectives’ on the quantum system. Each context C determines and is determined by a set of commuting self-adjoint operators, which physically correspond to co-measurable physical quantities. The trivial abelian C^* -algebra $\mathbb{C}\hat{1}$ represents the trivial classical perspective. We will not be concerned with physical interpretation in the following. In the second article [17], we will treat the time evolution of quantum systems.*

The poset $\mathcal{C}(\mathcal{A})_0$ has arbitrary meets (greatest lower bounds). For any family $(C_i)_{i \in I} \subseteq \mathcal{C}(\mathcal{A})_0$,

$$\bigwedge_{i \in I} C_i := \bigcap_{i \in I} C_i. \quad (1)$$

In $\mathcal{C}(\mathcal{A})$, not all meets exist. Two algebras $C_1, C_2 \in \mathcal{C}(\mathcal{A})$ have a meet if and only if they intersect not just in the trivial algebra. Both posets $\mathcal{C}(\mathcal{A})_0$ and $\mathcal{C}(\mathcal{A})$ have all directed joins, hence they are directed complete partial orders (dcpos). For any directed family $(C_i)_{i \in I} \subseteq \mathcal{C}(\mathcal{A})$, the directed join is given by the (abelian) C^* -algebra generated by the algebras C_i . It was shown in [18], Prop. 5.25 that there is a functor

$$\begin{aligned} \mathcal{C} : \mathbf{uC}^* &\longrightarrow \mathbf{Dcpo} \\ \mathcal{A} &\longmapsto \mathcal{C}(\mathcal{A}) \end{aligned} \quad (2)$$

from the category \mathbf{uC}^* of unital C^* -algebras and unital $*$ -homomorphisms to the category \mathbf{Dcpo} of dcpo and Scott-continuous functions.

If \mathcal{A} is abelian, then it is the top element of $\mathcal{C}(\mathcal{A})$ (and $\mathcal{C}(\mathcal{A})_0$ is a complete lattice). In this case, all joins exist in $\mathcal{C}(\mathcal{A})$ (respectively $\mathcal{C}(\mathcal{A})_0$). If \mathcal{A} is nonabelian, then the maximal abelian C^* -subalgebras are the maximal elements in the poset $\mathcal{C}(\mathcal{A})$ (respectively $\mathcal{C}(\mathcal{A})_0$), but no top element exists.

We now introduce the spectral presheaf of a unital C^* -algebra, which is a generalisation of the Gelfand spectrum of a unital abelian C^* -algebra.

Definition 2 *Let \mathcal{A} be a unital C^* -algebra \mathcal{A} . The spectral presheaf $\underline{\Sigma}$ of \mathcal{A} is the presheaf over $\mathcal{C}(\mathcal{A})$ given*

- (a) *on objects: for all $C \in \mathcal{C}(\mathcal{A})$, $\underline{\Sigma}_C := \Sigma(C)$, the Gelfand spectrum of C ,*
- (b) *on arrows: for all inclusions $i_{C'C} : C' \hookrightarrow C$,*

$$\begin{aligned} \underline{\Sigma}(i_{C'C}) : \underline{\Sigma}_C &\longrightarrow \underline{\Sigma}_{C'} \\ \lambda &\longmapsto \lambda|_{C'} \end{aligned} \tag{3}$$

*the canonical restriction map. This map is surjective and continuous with respect to the Gelfand topologies.*¹

Hence, the spectral presheaf $\underline{\Sigma}$ of a C^* -algebra \mathcal{A} consists of the Gelfand spectra of the abelian C^* -subalgebras of \mathcal{A} , ‘glued together’ in the canonical manner. If \mathcal{A} is abelian, then the component $\underline{\Sigma}_{\mathcal{A}}$ of $\underline{\Sigma}$ is the Gelfand spectrum of \mathcal{A} .

The spectral presheaf was first defined by Isham and Butterfield ([35], in [36] for von Neumann algebras) and used extensively in the so-called *topos approach* to quantum theory (see [25]). Being a presheaf, $\underline{\Sigma}$ is an object in the topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ of presheaves over the context category $\mathcal{C}(\mathcal{A})$.

Remark 3 *The poset $\mathcal{C}(\mathcal{A})$ (or $\mathcal{C}(\mathcal{A})_0$) can be equipped with different topologies. One of them is the lower Alexandroff topology for which all lower sets in $\mathcal{C}(\mathcal{A})$ are open sets. Order-preservation (monotonicity) of $\tilde{\phi}$ implies directly that the morphism $\tilde{\phi} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is continuous with respect to the Alexandroff topologies on $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$. Moreover, it is well-known that the topos of presheaves over $\mathcal{C}(\mathcal{A})$ is isomorphic to the topos of sheaves over $\mathcal{C}(\mathcal{A})_{\text{Alex}}$, the poset $\mathcal{C}(\mathcal{A})$ equipped with the Alexandroff topology,*

$$\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \simeq \mathbf{Sh}(\mathcal{C}(\mathcal{A})_{\text{Alex}}). \tag{4}$$

¹We recall that the Gelfand topology on the Gelfand spectrum $\Sigma(C)$ of a unital abelian C^* -algebra C is the relative weak*-topology on $\Sigma(C)$ seen as a subset of the dual C^* of C .

Another natural topology to consider is the Scott topology. As mentioned above, the morphism $\tilde{\phi} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is continuous with respect to the Scott topologies on $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$. In the following, we will not be concerned with topologies on $\mathcal{C}(\mathcal{A})$, hence we will discuss presheaves and not sheaves.

3 Algebra morphisms, geometric morphisms and maps between spectral presheaves

We present the basic construction showing that every unital $*$ -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ between unital C^* -algebras induces a map $\langle \Phi, \mathcal{G}_\phi \rangle : \underline{\Sigma}^{\mathcal{B}} \rightarrow \underline{\Sigma}^{\mathcal{A}}$ in the opposite direction between the spectral presheaves of the algebras. In the following sections, we will give some categorical background of this construction and will then focus on automorphisms of unital C^* -algebras and the corresponding automorphisms of their spectral presheaves.

Let $\mathcal{A}, \mathcal{B} \in \text{Ob}(\mathbf{uC}^*)$ be unital C^* -algebras, and let $\phi \in \mathbf{uC}^*(\mathcal{A}, \mathcal{B})$, that is, a unital $*$ -homomorphism from \mathcal{A} to \mathcal{B} . The algebra morphism ϕ induces a map

$$\begin{aligned} \tilde{\phi} : \mathcal{C}(\mathcal{A}) &\longrightarrow \mathcal{C}(\mathcal{B}) \\ C &\longmapsto \phi|_C(C) \end{aligned} \tag{5}$$

between the posets of unital abelian C^* -subalgebras of \mathcal{A} and \mathcal{B} , respectively. For $C \in \mathcal{C}(\mathcal{A})$, the image $\phi|_C(C)$ is norm-closed (and hence a C^* -algebra) since ϕ is a $*$ -homomorphism. The map $\tilde{\phi}$ preserves the partial order and hence is well-defined.

$\mathcal{C}(\mathcal{A})$ is the base category of the presheaf topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$, $\mathcal{C}(\mathcal{B})$ is the base category of $\mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}}$, and $\tilde{\phi} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$ is a morphism (i.e., functor) between these base categories. As is well-known, such a morphism between the base categories induces an essential geometric morphism

$$\Phi : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \longrightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}} \tag{6}$$

between the topoi. (For some background on geometric morphisms, see e.g. [39, 37]; for essential geometric morphisms, see in particular A.4.1.4 and 4.1.5 in [37].) $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ and $\mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}}$ are presheaf categories over small base categories, so every essential geometric morphism arises from a functor $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$. The geometric morphism Φ has a direct image part

$$\Phi_* : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \longrightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}} \tag{7}$$

in covariant direction, and an inverse image part

$$\begin{aligned}\Phi^* : \mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}} &\longrightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \\ \underline{P} &\longmapsto \underline{P} \circ \tilde{\phi}\end{aligned}\tag{8}$$

in contravariant direction. We display how the inverse image functor Φ^* acts on a presheaf here, since we will need this in the following (while the action of the direct image functor Φ_* will not play a role). Φ^* is left adjoint to Φ_* , and Φ^* preserves finite limits. Since Φ is an essential geometric morphism, Φ^* also has a left adjoint $\Phi_!$ in covariant direction, that is, $\Phi_! : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}}$. This implies that the inverse image functor Φ^* preserves all limits as well as colimits.

Let $\underline{\Sigma}^{\mathcal{B}}$ be the spectral presheaf of \mathcal{B} . This is an object in $\mathbf{Set}^{\mathcal{C}(\mathcal{B})^{\text{op}}}$. We use the inverse image functor Φ^* to map $\underline{\Sigma}^{\mathcal{B}}$ to an object $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$ in $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$: by (8), we have

$$\forall C \in \mathcal{C}(\mathcal{A}) : \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C = \underline{\Sigma}_{\tilde{\phi}(C)}^{\mathcal{B}}.\tag{9}$$

The restriction maps of the presheaf $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$ are given as follows: if $C', C \in \mathcal{C}(\mathcal{A})$ such that there is an inclusion $i_{C'C} : C' \hookrightarrow C$, then

$$\begin{aligned}\Phi^*(\underline{\Sigma}^{\mathcal{B}})(i_{C'C}) : \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C &\longrightarrow \Phi^*(\underline{\Sigma}^{\mathcal{B}})_{C'} \\ \lambda &\longmapsto \lambda|_{\phi(C')}.\end{aligned}\tag{10}$$

For each $C \in \mathcal{C}(\mathcal{A})$, we have a morphism

$$\phi|_C : C \longrightarrow \phi(C)\tag{11}$$

of unital abelian C^* -algebras. By Gelfand duality, this induces a continuous map

$$\begin{aligned}\mathcal{G}_{\phi;C} : \Sigma(\phi(C)) &\longrightarrow \Sigma(C) \\ \lambda &\longmapsto \lambda \circ \phi|_C\end{aligned}\tag{12}$$

in the opposite direction between the Gelfand spectra. Noting that $\Phi^*(\underline{\Sigma}^{\mathcal{B}})_C = \underline{\Sigma}_{\tilde{\phi}(C)}^{\mathcal{B}} = \Sigma(\phi(C))$ and $\underline{\Sigma}_C^{\mathcal{A}} = \Sigma(C)$, we have a map

$$\mathcal{G}_{\phi;C} : \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C \longrightarrow \underline{\Sigma}_C^{\mathcal{A}},\tag{13}$$

for each $C \in \mathcal{C}(\mathcal{A})$.

Let $C, C' \in \mathcal{C}(\mathcal{A})$ such that $C' \subset C$, and let $\lambda \in \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C$. Then

$$\mathcal{G}_{\phi;C'}(\Phi^*(\underline{\Sigma}^{\mathcal{B}})(i_{C'C})(\lambda)) = \mathcal{G}_{\phi;C'}(\lambda|_{\phi(C')}) \quad (14)$$

$$= \lambda|_{\phi(C')} \circ \phi|_{C'} \quad (15)$$

$$= (\lambda \circ \phi|_C)|_{C'} \quad (16)$$

$$= \underline{\Sigma}^{\mathcal{A}}(i_{C'C})(\lambda \circ \phi|_C) \quad (17)$$

$$= \underline{\Sigma}^{\mathcal{A}}(i_{C'C})(\mathcal{G}_{\phi;C}(\lambda)), \quad (18)$$

so

$$\mathcal{G}_{\phi;C'} \circ \Phi^*(\underline{\Sigma}^{\mathcal{B}})(i_{C'C}) = \underline{\Sigma}^{\mathcal{A}}(i_{C'C}) \circ \mathcal{G}_{\phi;C} \quad (19)$$

and the following diagram commutes for all $C', C \in \mathcal{C}(\mathcal{A})$ such that $C' \subset C$:

$$\begin{array}{ccc} \Phi^*(\underline{\Sigma}^{\mathcal{B}})_C & \xrightarrow{\mathcal{G}_{\phi;C}} & \underline{\Sigma}_C^{\mathcal{A}} \\ \Phi^*(\underline{\Sigma}^{\mathcal{B}})(i_{C'C}) \downarrow & & \downarrow \underline{\Sigma}^{\mathcal{A}}(i_{C'C}) \\ \Phi^*(\underline{\Sigma}^{\mathcal{B}})_{C'} & \xrightarrow{\mathcal{G}_{\phi;C'}} & \underline{\Sigma}_{C'}^{\mathcal{A}} \end{array}$$

This means that the maps $\mathcal{G}_{\phi;C}$, $C \in \mathcal{C}(\mathcal{A})$, are the components of a natural transformation

$$\mathcal{G}_{\phi} : \Phi^*(\underline{\Sigma}^{\mathcal{B}}) \longrightarrow \underline{\Sigma}^{\mathcal{A}}. \quad (20)$$

Thus, \mathcal{G}_{ϕ} is an arrow in the topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$, mapping the inverse image $\Phi^*(\underline{\Sigma}^{\mathcal{B}})$ of $\underline{\Sigma}^{\mathcal{B}}$, the spectral presheaf of \mathcal{B} , into $\underline{\Sigma}^{\mathcal{A}}$, the spectral presheaf of \mathcal{A} .

In a two-step process, we have mapped the spectral presheaf $\underline{\Sigma}^{\mathcal{B}}$ of \mathcal{B} into the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of \mathcal{A} ,

$$\underline{\Sigma}^{\mathcal{B}} \xrightarrow{\Phi^*} \Phi^*(\underline{\Sigma}^{\mathcal{B}}) \xrightarrow{\mathcal{G}_{\phi}} \underline{\Sigma}^{\mathcal{A}}. \quad (21)$$

Note that the map $\mathcal{G}_{\phi} \circ \Phi^* : \underline{\Sigma}^{\mathcal{B}} \rightarrow \underline{\Sigma}^{\mathcal{A}}$ is in contravariant direction with respect to the algebra morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ that we started from.

The ‘composite’ $\mathcal{G}_{\phi} \circ \Phi^*$ consists of the inverse image part of a geometric morphism between topoi, followed by an arrow in a topos. Hence, it is not a proper composite arrow in any category. But maps like $\mathcal{G}_{\phi} \circ \Phi^*$ —which, more conventionally, are also denoted $\langle \Phi, \mathcal{G}_{\phi} \rangle$ —are well-known in the theory of ringed topoi, where an arrow $\langle \Gamma, \eta \rangle$ is a geometric morphism $\Gamma : \mathcal{X} \rightarrow \mathcal{Y}$ between the topoi, together with a specified map (morphism of internal rings) $\eta : \Gamma^* R_{\mathcal{Y}} \rightarrow R_{\mathcal{X}}$ from the inverse image of the ring object $R_{\mathcal{Y}}$ in the ringed topos \mathcal{Y} to the ring object $R_{\mathcal{X}}$ in the ringed topos \mathcal{X} , see e.g. [40].

Summing up, we have shown:

Proposition 4 *Let \mathcal{A}, \mathcal{B} be two unital C^* -algebras, and let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital $*$ -homomorphism. There is a canonical map*

$$\langle \Phi, \mathcal{G}_\phi \rangle = \mathcal{G}_\phi \circ \Phi^* : \underline{\Sigma}^{\mathcal{B}} \longrightarrow \underline{\Sigma}^{\mathcal{A}} \quad (22)$$

in the opposite direction between the associated spectral presheaves.

4 Presheaves, copresheaves and ‘local duality’

In this section, we develop some categorical background to the construction presented in the previous section. This also relates the present work to work by Heunen, Landsman, Spitters and Wolters [32, 33, 34, 42]. There is a more refined categorical description using the fact that copresheaves are fibered over presheaves [28]. This will be developed in future work with Jonathon Funk, Pedro Resende and Rui Soares Barbosa.

4.1 Bohrification and partial C^* -algebras

The following construction is due to Heunen, Landsman and Spitters [32]:

Definition 5 *Let $\mathcal{A} \in \text{Ob}(\mathbf{uC}^*)$ be a unital C^* -algebra, and let $\mathcal{C}(\mathcal{A})$ be its context category. The Bohrification of \mathcal{A} is the (tautological) copresheaf $\overline{\mathcal{A}}$ over $\mathcal{C}(\mathcal{A})$ that is given*

(a) *on objects: $\forall C \in \mathcal{C}(\mathcal{A}) : \overline{\mathcal{A}}_C := C$,*

(b) *on arrows: for all inclusions $i_{C'C} : C' \hookrightarrow C$,*

$$\begin{aligned} \overline{\mathcal{A}}(i_{C'C}) : \overline{\mathcal{A}}_{C'} = C' &\longrightarrow \overline{\mathcal{A}}_C = C \\ \hat{A} &\longmapsto \hat{A}, \end{aligned} \quad (23)$$

that is, $\overline{\mathcal{A}}(i_{C'C}) = i_{C'C}$, and $\overline{\mathcal{A}}$ is the tautological presheaf over $\mathcal{C}(\mathcal{A})$.

The copresheaf $\overline{\mathcal{A}}$ is an object in the topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ of covariant functors from $\mathcal{C}(\mathcal{A})$ to \mathbf{Set} . Using results by Banaschewski and Mulvey [4, 5, 6, 7], one can show that $\overline{\mathcal{A}}$ is an *abelian* C^* -algebra internally in the topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$; for details, see [32, 33, 34].

Definition 6 Let \mathcal{A} be a unital C^* -algebra. The normal elements in \mathcal{A} , equipped with the involution inherited from \mathcal{A} and partial operations of addition and multiplication also inherited from \mathcal{A} , but defined only for (arbitrary pairs of) commuting elements, form the partial unital C^* -algebra associated with \mathcal{A} , which we will denote by $\mathcal{A}_{\text{part}}$.

Let \mathcal{A}, \mathcal{B} be unital C^* -algebras, and let $\mathcal{A}_{\text{part}}, \mathcal{B}_{\text{part}}$ be the associated unital partial C^* -algebras. A unital map

$$T : \mathcal{A}_{\text{part}} \rightarrow \mathcal{B}_{\text{part}} \quad (24)$$

such that for all commuting elements \hat{A}, \hat{B} of $\mathcal{A}_{\text{part}}$, we have

- (1) $T(\hat{A} + \hat{B}) = T(\hat{A}) + T(\hat{B})$,
- (2) $T(\hat{A}\hat{B}) = T(\hat{A})T(\hat{B})$,
- (3) $T((\hat{A}\hat{B})^*) = T(\hat{B})^*T(\hat{A})^* = (T(\hat{A}\hat{B}))^*$,

is called a morphism of unital partial C^* -algebras. A bijective morphism $T : \mathcal{A}_{\text{part}} \rightarrow \mathcal{A}_{\text{part}}$ from the unital partial C^* -algebra $\mathcal{A}_{\text{part}}$ to itself is called an automorphism. The automorphisms of $\mathcal{A}_{\text{part}}$ form a group $\text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}})$.

Condition (2) in the definition above implies $T(\hat{A})T(\hat{B}) = T(\hat{B})T(\hat{A})$, so T preserves commutativity. Condition (3) implies $T(\hat{A}^*) = T(\hat{A})^*$ for all $\hat{A} \in \mathcal{A}_{\text{part}}$ (by picking $\hat{B} = \hat{1}$). The fact that T preserves the involution implies that, for all unital abelian C^* -subalgebras $C \in \mathcal{C}(\mathcal{A})$, the image $T(C)$ is norm-closed and hence a unital abelian C^* -subalgebra of \mathcal{A} .

If $T \in \text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}})$ is an automorphism of the unital partial C^* -algebra $\mathcal{A}_{\text{part}}$, then (by definition) its inverse T^{-1} is an automorphism of $\mathcal{A}_{\text{part}}$, too.

Remark 7 The fact that the topos-internal C^* -algebra $\overline{\mathcal{A}}$ is abelian while the usual topos-external C^* -algebra \mathcal{A} may be non-abelian is due to the definition of the algebraic structure internally in the topos $\mathbf{Set}^{C(\mathcal{A})}$: basically speaking, one only considers algebraic operations within each abelian subalgebra $C \in \mathcal{C}(\mathcal{A})$, that is, only between commuting, normal operators. One simply ignores non-normal operators (which are not contained in any abelian C^* -subalgebra) and forgets addition and multiplication between non-commuting normal operators (which never lie in the same abelian subalgebra), and hence obtains an abelian algebra. Seen topos-externally, this algebra is the unital partial C^* -algebra $\mathcal{A}_{\text{part}}$. We will make use of this in section 5.

Van den Berg and Heunen discuss a number of aspects of partial C^* -algebras and their morphisms in [8].

4.2 Categories of presheaves and copresheaves and ‘local duality’

As we already saw, in order to discuss algebra morphisms between different (nonabelian) algebras and the corresponding morphisms between their spectral presheaves in the opposite direction, we need to consider presheaves and copresheaves over different base categories, since different algebras \mathcal{A}, \mathcal{B} have different context categories $\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{B})$ of abelian subalgebras.

The following construction of a category of presheaves (respectively copresheaves) with varying base categories was suggested by Nadish de Silva [41]:

Definition 8 *Let \mathcal{D} be a small category. The category $\mathbf{Presh}(\mathcal{D})$ of \mathcal{D} -valued presheaves has as its objects functors of the form $\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}$, where \mathcal{J} is a small category. Arrows are pairs*

$$\langle H, \eta \rangle : (\tilde{P} : \tilde{\mathcal{J}} \rightarrow \mathcal{D}^{\text{op}}) \longrightarrow (\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}), \quad (25)$$

where $H : \mathcal{J} \rightarrow \tilde{\mathcal{J}}$ is a functor and $\eta : H^* \tilde{P} \rightarrow \underline{P}$ is a natural transformation in $(\mathcal{D}^{\text{op}})^{\mathcal{J}}$. Here, $H^* \tilde{P}$ is the presheaf over \mathcal{J} given by

$$\forall J \in \mathcal{J} : H^* \tilde{P}_J = \tilde{P}_{H(J)}. \quad (26)$$

Let $\underline{P}_i : \mathcal{J}_i \rightarrow \mathcal{D}^{\text{op}}$, $i = 1, 2, 3$, be three presheaves over different base categories. Given two composable arrows $\langle H', \eta' \rangle : \underline{P}_3 \rightarrow \underline{P}_2$ and $\langle H, \eta \rangle : \underline{P}_2 \rightarrow \underline{P}_1$, the composite is $\langle H' \circ H, \eta \circ \eta' \rangle : \underline{P}_3 \rightarrow \underline{P}_1$, where, for all $J \in \mathcal{J}_1$, the natural transformation $\eta \circ \eta'$ has components

$$(\eta \circ \eta')_J = \eta_J \circ \eta'_{H(J)} : ((H' \circ H)^* \underline{P}_3)_J = (\hat{P}_3)_{H'(H(J))} \rightarrow (\underline{P}_2)_{H(J)} \rightarrow (\hat{P}_1)_J. \quad (27)$$

Every object $\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}$ in a presheaf category $\mathbf{Presh}(\mathcal{D})$ can be identified with a functor $\underline{P} : \mathcal{J}^{\text{op}} \rightarrow \mathcal{D}$ and hence can also be regarded as an object in the category $\mathcal{D}^{\mathcal{J}^{\text{op}}}$. We will assume that the small category \mathcal{D} embeds into \mathbf{Set} (that is, there is a forgetful functor $\mathcal{D} \rightarrow \mathbf{Set}$), so we can think of \underline{P} as an object in the topos $\mathbf{Set}^{\mathcal{J}^{\text{op}}}$, too.

If we interpret a morphism $\langle H, \eta \rangle : \tilde{P} \rightarrow \underline{P}$ between presheaves $\tilde{P} : \tilde{\mathcal{J}}^{\text{op}} \rightarrow \mathcal{D}$ and $\underline{P} : \mathcal{J}^{\text{op}} \rightarrow \mathcal{D}$ as a morphism between the presheaf topoi $\mathbf{Set}^{\tilde{\mathcal{J}}^{\text{op}}}$ and $\mathbf{Set}^{\mathcal{J}^{\text{op}}}$, then it consists of the inverse image part H^* of the essential geometric morphism $H : \mathbf{Set}^{\mathcal{J}^{\text{op}}} \rightarrow \mathbf{Set}^{\tilde{\mathcal{J}}^{\text{op}}}$ induced by the functor $H : \mathcal{J} \rightarrow \tilde{\mathcal{J}}$ between the base categories, and a natural transformation $\eta : H^* \tilde{P} \rightarrow \underline{P}$. Note that η behaves contravariantly with respect to H .

In this way, $\mathbf{Presh}(\mathcal{D})$ extends to a category of presheaf topoi, with the particular kind of morphisms between the objects in the presheaf topoi described above.

Analogously, we define:

Definition 9 *Let \mathcal{C} be a small category. The category $\mathbf{Copresh}(\mathcal{C})$ of \mathcal{C} -valued copresheaves has as its objects functors of the form $\overline{Q} : \mathcal{J} \rightarrow \mathcal{C}$, where \mathcal{J} is a small category. Arrows are pairs*

$$\langle I, \theta \rangle : (\overline{Q} : \mathcal{J} \rightarrow \mathcal{C}) \longrightarrow (\tilde{\overline{Q}} : \tilde{\mathcal{J}} \rightarrow \mathcal{C}), \quad (28)$$

where $I : \mathcal{J} \rightarrow \tilde{\mathcal{J}}$ is a functor and $\theta : \overline{Q} \rightarrow I^* \tilde{\overline{Q}}$ is a natural transformation in $\mathcal{C}^{\mathcal{J}}$. Here, $I^* \tilde{\overline{Q}}$ is the copresheaf over \mathcal{J} given by

$$\forall J \in \mathcal{J} : I^* \tilde{\overline{Q}}_J = \tilde{\overline{Q}}_{I(J)}. \quad (29)$$

Let $\overline{Q}_i : \mathcal{J}_i \rightarrow \mathcal{D}^{\text{op}}$, $i = 1, 2, 3$, be three copresheaves over different base categories. Given two composable arrows $\langle I, \theta \rangle : \overline{Q}_1 \rightarrow \overline{Q}_2$ and $\langle I', \theta' \rangle : \overline{Q}_2 \rightarrow \overline{Q}_3$, the composite is $\langle I' \circ I, \theta' \circ \theta \rangle : \overline{Q}_1 \rightarrow \overline{Q}_3$, where, for all $J \in \mathcal{J}_1$, the natural transformation $\theta' \circ \theta$ has components

$$(\theta' \circ \theta)_J = \theta'_{I'(J)} \circ \theta_J : (\overline{Q}_1)_J \rightarrow (\overline{Q}_2)_{I(J)} \rightarrow (\overline{Q}_3)_{I'(I(J))} = ((I' \circ I)^* (\overline{Q}_3))_J. \quad (30)$$

We assume that there is a forgetful functor from \mathcal{C} to \mathbf{Set} . Then the category $\mathbf{Copresh}(\mathcal{C})$ extends to a category of copresheaf topoi, with the morphisms $\langle I, \theta \rangle : \overline{Q} \rightarrow \tilde{\overline{Q}}$ between copresheaves $\overline{Q} : \mathcal{J} \rightarrow \mathcal{C}$ and $\tilde{\overline{Q}} : \tilde{\mathcal{J}} \rightarrow \mathcal{C}$ in different copresheaf topoi given by the inverse image part I^* of the essential geometric morphism $I : \mathbf{Set}^{\mathcal{J}} \rightarrow \mathbf{Set}^{\tilde{\mathcal{J}}}$ induced by the functor $I : \mathcal{J} \rightarrow \tilde{\mathcal{J}}$, and a natural transformation $\theta : \overline{Q} \rightarrow I^* \tilde{\overline{Q}}$. For copresheaves, the natural transformation θ behaves covariantly with respect to I .

We note that the construction of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ is based on ‘local’ Gelfand duality: for each context $C \in \mathcal{C}(\mathcal{A})$, the component $\underline{\Sigma}_C^{\mathcal{A}}$ is simply given by the Gelfand spectrum of C . Hence, we use standard Gelfand duality locally in each abelian part and glue the Gelfand spectra together into a presheaf in the canonical way. Hence, for each $C \in \mathcal{C}(\mathcal{A})$, the component $\underline{\Sigma}_C^{\mathcal{A}}$ of the spectral presheaf is the spectrum of the component $\overline{\mathcal{A}}_C = C$ of the copresheaf $\overline{\mathcal{A}}$.

We now want to formalise this situation of having a ‘local duality’ between a copresheaf (e.g. of algebras) and a presheaf (e.g. of their spectra).

Proposition 10 *Let \mathcal{C}, \mathcal{D} be two small categories that are dually equivalent,*

$$\mathcal{C} \begin{array}{c} \xrightarrow{f} \\ \xleftarrow[\perp]{g} \end{array} \mathcal{D}^{\text{op}}. \quad (31)$$

Then there is a dual equivalence

$$\mathbf{Copresh}(\mathcal{C}) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow[\perp]{G} \end{array} \mathbf{Presh}(\mathcal{D})^{\text{op}}. \quad (32)$$

Proof. We first show that $g : \mathcal{D}^{\text{op}} \rightarrow \mathcal{C}$ induces a functor $G : \mathbf{Presh}(\mathcal{D})^{\text{op}} \rightarrow \mathbf{Copresh}(\mathcal{C})$. On objects, G is given as follows: let $\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}$, then $G(\underline{P}) : \mathcal{J} \rightarrow \mathcal{C}$ is defined by

- (a) $\forall J \in \mathcal{J} : G(\underline{P})_J := g(\underline{P}_J),$
- (b) $\forall a : J' \rightarrow J : G(a) := g(\underline{P}(a)) : G(\underline{P})_{J'} \rightarrow G(\underline{P})_J.$

On arrows: let $\langle H, \eta \rangle : (\tilde{P} : \tilde{\mathcal{J}} \rightarrow \mathcal{D}^{\text{op}}) \rightarrow (\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}})$ be an arrow in $\mathbf{Presh}(\mathcal{D})$, then

$$G(\langle H, \eta \rangle) := \langle H, \theta \rangle, \quad (33)$$

where $\theta : G(\underline{P}) \rightarrow H^*(G(\tilde{P}))$ is given, for all $J \in \mathcal{J}$, by

$$\theta_J := g(\eta_J) : G(\underline{P})_J \longrightarrow H^*(G(\tilde{P}))_J. \quad (34)$$

(Note that $\eta_J : (H^*\tilde{P})_J = \tilde{P}_{H(J)} \rightarrow \underline{P}_J$, so $g(\eta_J) : g(\underline{P}_J) = G(\underline{P})_J \rightarrow g(\tilde{P}_{H(J)}) = G(\tilde{P})_{H(J)} = H^*(G(\tilde{P}))_J$.)

Analogously, $f : \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$ induces $F : \mathbf{Copresh}(\mathcal{C}) \rightarrow \mathbf{Presh}(\mathcal{D})$: let $\overline{Q} : \mathcal{J} \rightarrow \mathcal{C}$, then $F(\overline{Q}) : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}$ is defined

- (a) $\forall J \in \mathcal{J} : F(\overline{Q})_J := f(\overline{Q}_J),$
- (b) $\forall a : J' \rightarrow J : F(a) := f(\overline{Q}(a)) : F(\overline{Q})_{J'} \rightarrow F(\overline{Q})_J.$

On arrows: let $\langle I, \theta \rangle : (\overline{Q} : \mathcal{J} \rightarrow \mathcal{C}) \rightarrow (\tilde{Q} : \tilde{\mathcal{J}} \rightarrow \mathcal{C})$ be an arrow in $\mathbf{Copresh}(\mathcal{C})$, then

$$F(\langle I, \theta \rangle) := \langle I, \eta \rangle, \quad (35)$$

where $\eta : I^*(F(\tilde{Q})) \rightarrow F(\overline{Q})$ is given, for all $J \in \mathcal{J}$, by

$$\eta_J := f(\theta_J) : I^*(F(\tilde{Q}))_J \longrightarrow F(\overline{Q})_J. \quad (36)$$

(Note that $\theta_J : \overline{Q}_J \rightarrow (I^*\overline{Q})_J = \overline{Q}_{I(J)}$, so $f(\theta_J) : f(\overline{Q}_{I(J)}) = F(\overline{Q})_{I(J)} = I^*(F(\overline{Q}))_J \rightarrow f(\overline{Q}_J) = F(\overline{Q})_J$.)

Since $f \dashv g$, we have natural transformations $\epsilon : f \circ g \rightarrow \text{id}_{\mathcal{D}^{\text{op}}}$ (the counit of the adjunction) and $\eta : \text{id}_{\mathcal{C}} \rightarrow g \circ f$ (the unit of the adjunction).

Let $\underline{P} : \mathcal{J} \rightarrow \mathcal{D}^{\text{op}}$ be a \mathcal{D} -valued presheaf over \mathcal{J} . By definition, we have, for each $J \in \mathcal{J}$,

$$(F(G(\underline{P})))_J = f(G(\underline{P})_J) = f(g(\underline{P}_J)), \quad (37)$$

and naturality of $\epsilon : f \circ g \rightarrow \text{id}_{\mathcal{D}^{\text{op}}}$ guarantees that $F \circ G : \mathbf{Presh}(\mathcal{D}) \rightarrow \mathbf{Presh}(\mathcal{D})$ is a morphism, namely $F \circ G = \langle \text{id}_{\mathcal{D}^{\text{op}}}, \epsilon \rangle$. Analogously, $G \circ F = \langle \text{id}_{\mathcal{C}}, \eta \rangle : \mathbf{Copresh}(\mathcal{C}) \rightarrow \mathbf{Copresh}(\mathcal{C})$. ■

In this kind of duality between \mathcal{D} -valued presheaves and \mathcal{C} -valued copresheaves using ‘local duality’ between \mathcal{D}^{op} and \mathcal{C} , there are two reversals of direction: one in the variance of the functors (presheaves and copresheaves), the other in the direction of morphisms between them.

4.3 Application to Gelfand duality

Gelfand duality is the dual equivalence

$$\mathbf{ucC}^* \begin{array}{c} \xrightarrow{\Sigma} \\ \perp \\ \xleftarrow{C(-)} \end{array} \mathbf{KHaus}^{\text{op}}. \quad (38)$$

between unital abelian C^* -algebras and compact Hausdorff spaces. This gives a dual equivalence

$$\mathbf{Copresh}(\mathbf{ucC}^*) \begin{array}{c} \xrightarrow{\Sigma} \\ \perp \\ \xleftarrow{C(-)} \end{array} \mathbf{Presh}(\mathbf{KHaus})^{\text{op}}. \quad (39)$$

Given a (generally *nonabelian*) unital C^* -algebra \mathcal{A} , we can define its context category $\mathcal{C}(\mathcal{A})$, i.e., the directed complete partial order of abelian C^* -subalgebras that share the unit element with \mathcal{A} . The Bohrification $\overline{\mathcal{A}} : \mathcal{C}(\mathcal{A}) \rightarrow \mathbf{ucC}^*$, given by $\overline{\mathcal{A}}_C = C$ for all $C \in \mathcal{A}$, is an object in the category $\mathbf{Copresh}(\mathbf{ucC}^*)$. The spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of \mathcal{A} , given by $\underline{\Sigma}_C^{\mathcal{A}} = \Sigma(C)$, the Gelfand spectrum of C , is an object in $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$.

These two objects correspond to each other via (39),

$$\Sigma(\overline{\mathcal{A}}) = \underline{\Sigma}^{\mathcal{A}} : \mathcal{C}(\mathcal{A}) \longrightarrow \mathbf{KHaus}^{\text{op}} \quad (40)$$

and

$$C(\underline{\Sigma}^{\mathcal{A}}) = \overline{\mathcal{A}} : \mathcal{C}(\mathcal{A}) \longrightarrow \mathbf{ucC}^*. \quad (41)$$

If \mathcal{A}, \mathcal{B} are two (nonabelian) unital C^* -algebras and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a unital $*$ -homomorphism, then we obtain a monotone map

$$\begin{aligned} \tilde{\phi} : \mathcal{C}(\mathcal{A}) &\longrightarrow \mathcal{C}(\mathcal{B}) \\ C &\longmapsto \phi(C). \end{aligned} \quad (42)$$

Define a natural transformation $\theta : \overline{\mathcal{A}} \rightarrow \tilde{\phi}^* \overline{\mathcal{B}}$ by

$$\forall C \in \mathcal{C}(\mathcal{A}) : \theta_C := \phi|_C : \overline{\mathcal{A}}_C = C \longrightarrow (\tilde{\phi}^* \overline{\mathcal{B}})_C = \overline{\mathcal{B}}_{\phi(C)} = \phi(C). \quad (43)$$

Then

$$\langle \tilde{\phi}, \theta \rangle : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{B}} \quad (44)$$

is a morphism in $\mathbf{Copresh}(\mathbf{ucC}^*)$. This is essentially Prop. 34 in [8] (see version 3).²

Dually, define a natural transformation $\eta : \tilde{\phi}^* \underline{\Sigma}^{\mathcal{B}} \rightarrow \underline{\Sigma}^{\mathcal{A}}$ by

$$\forall C \in \mathcal{C}(\mathcal{A}) : \eta_C := \Sigma(\phi|_C) : (\tilde{\phi}^* \underline{\Sigma}^{\mathcal{B}})_C = \underline{\Sigma}_{\phi(C)}^{\mathcal{B}} = \Sigma(\phi(C)) \longrightarrow \underline{\Sigma}_C^{\mathcal{A}} = \Sigma(C). \quad (45)$$

Then

$$\langle \tilde{\phi}, \eta \rangle : \underline{\Sigma}^{\mathcal{B}} \longrightarrow \underline{\Sigma}^{\mathcal{A}} \quad (46)$$

is a morphism in $\mathbf{Presh}(\mathbf{KHaus})$.

By construction, $\Sigma(\langle \tilde{\phi}, \theta \rangle) = \langle \tilde{\phi}, \eta \rangle$ and $C(\langle \tilde{\phi}, \eta \rangle) = \langle \tilde{\phi}, \theta \rangle$ (using the duality in (39)).

Summing up, we have shown:

Theorem 11 *There is a contravariant functor*

$$\underline{\mathcal{S}} : \mathbf{ucC}^* \longrightarrow \mathbf{Presh}(\mathbf{KHaus}) \quad (47)$$

from the category of unital C^ -algebras to the category of compact Hausdorff space-valued presheaves, given*

(a) *on objects: $\forall \mathcal{A} \in \mathbf{Ob}(\mathbf{ucC}^*) : \underline{\mathcal{S}}(\mathcal{A}) := \underline{\Sigma}^{\mathcal{A}}$, the spectral presheaf of \mathcal{A} ,*

²Van den Berg and Heunen make a different choice for the direction of their geometric morphism, which leads to an arrow between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ in the opposite direction to the morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$. Our choice, which gives an arrow in the same direction as the external morphism ϕ , seems more natural.

- (b) on arrows: for all unital $*$ -morphisms $(\phi : \mathcal{A} \rightarrow \mathcal{B}) \in \text{Arr}(\mathbf{uC}^*)$,
 $\underline{\mathcal{S}}(\phi) = \langle \tilde{\phi}, \eta \rangle : \underline{\mathcal{S}}(\mathcal{B}) \rightarrow \underline{\mathcal{S}}(\mathcal{A})$.

Moreover, there is a covariant functor

$$\overline{\mathcal{B}} : \mathbf{uC}^* \longrightarrow \mathbf{Copresh}(\mathbf{uC}^*) \quad (48)$$

from the category of unital C^* -algebras to the category of unital abelian C^* -algebra-valued copresheaves, given

- (c) on objects: $\forall \mathcal{A} \in \text{Ob}(\mathbf{uC}^*) : \overline{\mathcal{B}}(\mathcal{A}) := \overline{\mathcal{A}}$, the Bohrification of \mathcal{A} ,
 (d) on arrows: for all unital $*$ -morphisms $(\phi : \mathcal{A} \rightarrow \mathcal{B}) \in \text{Arr}(\mathbf{uC}^*)$,
 $\overline{\mathcal{B}}(\phi) = \langle \tilde{\phi}, \theta \rangle : \overline{\mathcal{B}}(\mathcal{A}) \rightarrow \overline{\mathcal{B}}(\mathcal{B})$.

For each $\mathcal{A} \in \text{Ob}(\mathbf{uC}^*)$, the object $\overline{\mathcal{B}}(\mathcal{A}) = \overline{\mathcal{A}}$ is an object in the category $\mathbf{uC}^{*\mathcal{C}(\mathcal{A})}$, which is a subcategory of the copresheaf topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$. By the ‘local duality’ described in (39), $\overline{\mathcal{A}}$ corresponds to $\underline{\Sigma}^{\mathcal{A}}$, which is an object in the category $\mathbf{KHaus}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ that is a subcategory of the presheaf topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$.

As mentioned above, $\overline{\mathcal{A}}$ is an internal abelian C^* -algebra in $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$, as was shown in [32]. Morphisms between presheaves (respectively copresheaves) with different base categories $\mathcal{J}, \tilde{\mathcal{J}}$, but the same codomain category \mathcal{D} (respectively \mathcal{C}) are of the kind described in Def. 8 (respectively Def. 9).

Thm. 11 should be compared with section 3; in particular, the existence of the functor $\underline{\mathcal{S}}$ subsumes the result of Prop. 4.

5 Automorphisms of the spectral presheaf

From here on, we will focus on automorphisms of the spectral presheaf, and on their relation to certain automorphisms of unital C^* -algebras and von Neumann algebras.

We will first consider von Neumann algebras in subsection 5.1, for which the following characterisation is possible: if \mathcal{N} has no type I_2 -summand, then the group of automorphisms of the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$ of a von Neumann algebra \mathcal{N} is contravariantly isomorphic to the group of Jordan $*$ -automorphisms of \mathcal{N} . Moreover, we will show that the following groups are isomorphic: the group of Jordan $*$ -automorphisms of a von Neumann algebra \mathcal{N} , the group of automorphisms of the partial von Neumann algebra $\mathcal{N}_{\text{part}}$ associated with \mathcal{N} , the group of automorphisms of the complete orthomodular lattice $\mathcal{P}(\mathcal{N})$ of projections in \mathcal{N} (by a theorem by Dye), and the group

of order-automorphisms of the poset $\mathcal{V}(\mathcal{N})$ of abelian von Neumann subalgebras of \mathcal{N} . This is the content of Thm. 19, which is one of the main results of this paper.

In subsection 5.2, we will characterise automorphisms of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of a unital C^* -algebra \mathcal{A} . We show that if \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$, then the group of automorphisms of $\underline{\Sigma}^{\mathcal{A}}$ is contravariantly isomorphic to the group of automorphisms of the unital partial C^* -algebra \mathcal{A}_{part} associated with \mathcal{A} , to the group of quasi-Jordan automorphisms of the Jordan algebra \mathcal{A}_{sa} , and, using a recent result by Hamhalter, to the group of order-automorphisms of the poset $\mathcal{C}(\mathcal{A})$ of unital abelian C^* -subalgebras of \mathcal{A} . This is the content of Thm. 30.

Using a further result by Hamhalter, we show that for a certain class of C^* -algebras, the group of automorphisms of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ is contravariantly isomorphic to the group of Jordan $*$ -automorphisms of \mathcal{A} , thus generalising the result for von Neumann algebras to a larger class of algebras, see Cor. 33.

Moreover, we show that there is an injective contravariant group homomorphism from the automorphism group of a unital C^* -algebra \mathcal{A} into the group $\text{Aut}(\underline{\Sigma}^{\mathcal{A}})$ of automorphisms of its spectral presheaf, see Prop. 26. This directly implies the analogous result for von Neumann algebras.

5.1 Von Neumann algebras

Definition 12 *Let \mathcal{N} be a von Neumann algebra, and let $\mathcal{V}(\mathcal{N})$ denote the poset of non-trivial abelian von Neumann subalgebras that share the unit element with \mathcal{N} . $\mathcal{V}(\mathcal{N})$ is called the context category of \mathcal{N} . The normal elements in \mathcal{N} , equipped with the involution inherited from \mathcal{N} and partial operations of addition and multiplication also inherited from \mathcal{N} , but defined only for commuting elements, is the partial von Neumann algebra associated with \mathcal{N} , which we will denote \mathcal{N}_{part} .*

Let \mathcal{M}, \mathcal{N} be von Neumann algebras, and let $\mathcal{M}_{part}, \mathcal{N}_{part}$ be the associated partial von Neumann algebras. A unital map

$$T : \mathcal{M}_{part} \longrightarrow \mathcal{N}_{part} \tag{49}$$

that is normal, that is, ultraweakly continuous on (ultraweakly closed) commuting subsets of \mathcal{M}_{part} , and such that for all commuting elements \hat{A}, \hat{B} of \mathcal{M}_{part} , we have

$$(1) \quad T(\hat{A} + \hat{B}) = T(\hat{A}) + T(\hat{B}),$$

- (2) $T(\hat{A}\hat{B}) = T(\hat{A})T(\hat{B})$,
(3) $T((\hat{A}\hat{B})^*) = T(\hat{B})^*T(\hat{A})^* = (T(\hat{A}\hat{B}))^*$,

is called a morphism of partial von Neumann algebras. A bijective morphism $T : \mathcal{N}_{part} \rightarrow \mathcal{N}_{part}$ from the partial von Neumann algebra \mathcal{N}_{part} to itself is called an automorphism. The automorphisms of \mathcal{N}_{part} form a group $\text{Aut}_{part}(\mathcal{N}_{part})$.

The second condition in the definition above implies that T preserves commutativity, the third that it preserves the involution. The fact that T is normal on commuting subsets implies that, for all abelian von Neumann subalgebras $V \in \mathcal{V}(\mathcal{M})$, the image $T(V)$ is an abelian von Neumann subalgebra of \mathcal{N} . The inverse T^{-1} of an automorphism T of the partial von Neumann algebra \mathcal{N}_{part} is an automorphism of \mathcal{N}_{part} , too.

Definition 13 Let \mathcal{N} be a von Neumann algebra, and let $\underline{\Sigma}^{\mathcal{N}}$ be its spectral presheaf. An automorphism of $\underline{\Sigma}^{\mathcal{N}}$ is a pair $\langle \Gamma, \eta \rangle$, where $\Gamma : \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}} \rightarrow \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}}$ is an essential geometric automorphism, induced by an order-automorphism $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ (called the twisting map). $\Gamma^* : \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}} \rightarrow \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}}$ is the inverse image functor of the geometric automorphism Γ , and $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{N}}) \rightarrow \underline{\Sigma}^{\mathcal{N}}$ is a natural isomorphism for which each component $\eta_V : (\Gamma^*(\underline{\Sigma}^{\mathcal{N}}))_V \rightarrow \underline{\Sigma}_V^{\mathcal{N}}$, $V \in \mathcal{V}(\mathcal{N})$, is a homeomorphism. Hence, an automorphism $\langle \Gamma, \eta \rangle$ acts by

$$\underline{\Sigma}^{\mathcal{N}} \xrightarrow{\Gamma^*} \Gamma^*(\underline{\Sigma}^{\mathcal{N}}) \xrightarrow{\eta} \underline{\Sigma}^{\mathcal{N}}. \quad (50)$$

We will also use the notation $\eta \circ \Gamma^*$ for an automorphism $\langle \Gamma, \eta \rangle$ (although this ‘composition’ is not a composition of morphisms in a single category, but the inverse image part of a geometric automorphism, followed by a natural transformation).

Lemma 14 The automorphisms of $\underline{\Sigma}^{\mathcal{N}}$ form a group $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})$.

Proof. We define the group operation as

$$\begin{aligned} \text{Aut}(\underline{\Sigma}^{\mathcal{N}}) \times \text{Aut}(\underline{\Sigma}^{\mathcal{N}}) &\longrightarrow \text{Aut}(\underline{\Sigma}^{\mathcal{N}}) \\ (\langle \Gamma_1, \eta_1 \rangle, \langle \Gamma_2, \eta_2 \rangle) &\longmapsto \langle \Gamma_2 \circ \Gamma_1, \eta_1 \circ \eta_2 \rangle, \end{aligned} \quad (51)$$

where $\Gamma_2 \circ \Gamma_1 : \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}} \rightarrow \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{op}}$ is the essential geometric automorphism induced by $\gamma_2 \circ \gamma_1$, the composite of the twisting maps underlying Γ_2 respectively Γ_1 , and $\eta_1 \circ \eta_2$ is the natural isomorphism with components

$$\forall V \in \mathcal{V}(\mathcal{N}) : (\eta_1 \circ \eta_2)_V = \eta_{1;V} \circ \eta_{2;\gamma_1(V)} \quad (52)$$

(cf. Def. 8). The automorphism $\langle \text{Id}, \text{id} \rangle$ acts as a neutral element for the group operation. If $\langle \Gamma, \eta \rangle$ is an automorphism of $\underline{\Sigma}^{\mathcal{N}}$ with underlying twisting map $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$, then $\langle \Gamma^{-1}, \eta^{-1} \rangle$ with underlying twisting map γ^{-1} is its inverse. ■

An automorphism of $\underline{\Sigma}^{\mathcal{N}}$ as an object in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\text{op}}}$ in the sense of Def. 13 corresponds to an automorphism of $\underline{\Sigma}^{\mathcal{N}}$ as an object in the category $\mathbf{Presh}(\mathbf{KHaus})$, see Def. 8. Conversely, each automorphism of $\underline{\Sigma}^{\mathcal{N}}$ in $\mathbf{Presh}(\mathbf{KHaus})$ determines a unique automorphism of $\underline{\Sigma}^{\mathcal{N}}$ as an object of $\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\text{op}}}$.

Note that instead of requiring that the twisting map $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ is an order-automorphism, we could equivalently require γ is an invertible covariant functor. Even more concisely, we could suppress any reference to the twisting map and just require that there is an essential geometric automorphism $\Gamma : \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\text{op}}}$: each essential geometric morphism between topoi is induced by a (covariant) functor between the base categories of the topoi, that is, a functor $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ in our case. Moreover, the fact that Γ is a geometric automorphism implies that the functor γ must be invertible.

The action of Γ^* on $\underline{\Sigma}^{\mathcal{A}}$ —and on any other presheaf in the topos—can be seen as a ‘re-indexing’ or ‘twisting’. Concretely, the component of $\Gamma^*(\underline{\Sigma}^{\mathcal{N}})$ at $V \in \mathcal{V}(\mathcal{N})$ is given by $\Gamma^*(\underline{\Sigma}^{\mathcal{N}})_V = \underline{\Sigma}_{\gamma(V)}^{\mathcal{N}}$, so we assign the Gelfand spectrum of $\gamma(V)$ to V .

Note that if and only if V and $\gamma(V)$ are isomorphic as abelian C^* -algebras, there is a homeomorphism (i.e., an isomorphism in the category of topological spaces) $\eta_V : \underline{\Sigma}_{\gamma(V)}^{\mathcal{N}} \rightarrow \underline{\Sigma}_V^{\mathcal{N}}$ between their spectra. The definition of an automorphism of $\underline{\Sigma}^{\mathcal{N}}$ requires that such an isomorphism η_V exists for every $V \in \mathcal{V}(\mathcal{N})$, and that the η_V are the components of a natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{N}}) \rightarrow \underline{\Sigma}^{\mathcal{N}}$. We will see in Thm. 19 that in fact, given γ , the existence and uniqueness of η are guaranteed: every order-automorphism $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ induces a unique automorphism $\langle \Gamma, \eta \rangle$ of the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$, and in particular, a unique natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{N}}) \rightarrow \underline{\Sigma}^{\mathcal{N}}$.

We present two results that actually will be proven in subsection 5.2 for the more general case of unital C^* -algebras, which is why we do not give the proofs here.

Proposition 15 *Let \mathcal{N} be a von Neumann algebra. There is a contravariant group isomorphism*

$$A : \text{Aut}(\underline{\Sigma}^{\mathcal{N}}) \longrightarrow \text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}}) \quad (53)$$

between $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})$, the group of automorphisms of the spectral presheaf of \mathcal{N} , and $\text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}})$, the group automorphisms of the partial von Neumann algebra $\mathcal{N}_{\text{part}}$.

This result follows straightforwardly from the analogous result for unital C^* -algebras, see Prop. 23 below (using Lemmas 21 and 22).

Proposition 16 *Let \mathcal{N} be a von Neumann algebra, and let $\underline{\Sigma}^{\mathcal{N}}$ be its spectral presheaf. Let $\text{Aut}(\mathcal{N})$ be the automorphism group of \mathcal{N} , and let $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})$ be the automorphism group of $\underline{\Sigma}^{\mathcal{N}}$. There is an injective group homomorphism from $\text{Aut}(\mathcal{N})$ to $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})^{\text{op}}$ (that is, there is an injective, contravariant group homomorphism from $\text{Aut}(\mathcal{N})$ into $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})$), given by*

$$\begin{aligned} \text{Aut}(\mathcal{N}) &\longrightarrow \text{Aut}(\underline{\Sigma}^{\mathcal{N}})^{\text{op}} \\ \phi &\longmapsto \langle \Phi, \mathcal{G}_{\phi} \rangle = \mathcal{G}_{\phi} \circ \Phi^* \end{aligned} \quad (54)$$

This follows easily from Prop. 26, the analogous result for unital C^* -algebras.

Proposition 17 *Let \mathcal{N} be a von Neumann algebra, and let $\mathcal{N}_{\text{part}}$ be the corresponding partial von Neumann algebra. There is a group isomorphism*

$$B : \text{Aut}_{\text{cOML}}(\mathcal{P}(\mathcal{N})) \longrightarrow \text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}}) \quad (55)$$

between the group $\text{Aut}_{\text{cOML}}(\mathcal{P}(\mathcal{N}))$ of automorphisms $\tilde{T} : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{N})$ of the complete orthomodular lattice $\mathcal{P}(\mathcal{N})$ of projections in \mathcal{N} and the group $\text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}})$ of automorphisms $T : \mathcal{N}_{\text{part}} \rightarrow \mathcal{N}_{\text{part}}$ of the partial von Neumann algebra $\mathcal{N}_{\text{part}}$.

Proof. Let $T : \mathcal{N}_{\text{part}} \rightarrow \mathcal{N}_{\text{part}}$ be a partial von Neumann automorphism. Then, for all projections $\hat{P} \in \mathcal{P}(\mathcal{N}) \subset \mathcal{N}_{\text{part}}$,

$$T(\hat{P}) = T(\hat{P}^2) = T(\hat{P})^2, \quad T(\hat{P}) = T(\hat{P}^*) = T(\hat{P})^*, \quad (56)$$

so $T(\hat{P})$ is a projection, and $T : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{N})$ is a bijection. If $\hat{P} \leq \hat{Q}$, then

$$T(\hat{P}) = T(\hat{P}\hat{Q}) = T(\hat{P})T(\hat{Q}), \quad (57)$$

so $T(\hat{P}) \leq T(\hat{Q})$, that is, T preserves the order. Let $T^{-1} : \mathcal{N}_{\text{part}} \rightarrow \mathcal{N}_{\text{part}}$ be the inverse map of T , which is an automorphism of the partial von Neumann algebra $\mathcal{N}_{\text{part}}$, too. Then, if $\hat{P} \leq \hat{Q}$,

$$T^{-1}(\hat{P}) = T^{-1}(\hat{P}\hat{Q}) = T^{-1}(\hat{P})T^{-1}(\hat{Q}), \quad (58)$$

so $T^{-1}(\hat{P}) \leq T^{-1}(\hat{Q})$, and T reflects the order. Moreover, for all $\hat{P} \in \mathcal{P}(\mathcal{N})$,

$$T(\hat{1} - \hat{P}) = T(\hat{1}) - T(\hat{P}) = \hat{1} - T(\hat{P}), \quad (59)$$

so T preserves complements. Let $(\hat{P}_i)_{i \in I}$ be a family of projections, not necessarily commuting. Then

$$\forall i \in I : T(\hat{P}_i) \leq T\left(\bigvee_{i \in I} \hat{P}_i\right), \quad (60)$$

so

$$\bigvee_{i \in I} T(\hat{P}_i) \leq T\left(\bigvee_{i \in I} \hat{P}_i\right). \quad (61)$$

Note that here, we use joins in $\mathcal{P}(\mathcal{N})$ between not necessarily commuting projections, so we need the lattice structure of $\mathcal{P}(\mathcal{N})$, not just the partial algebra structure of \mathcal{N}_{part} . Since T^{-1} also preserves and reflects the order, (61) is equivalent to

$$T^{-1}\left(\bigvee_{i \in I} T(\hat{P}_i)\right) \leq \bigvee_{i \in I} \hat{P}_i. \quad (62)$$

For the left hand side, we have

$$T^{-1}\left(\bigvee_{i \in I} T(\hat{P}_i)\right) \geq \bigvee_{i \in I} T^{-1}(T(\hat{P}_i)) = \bigvee_{i \in I} \hat{P}_i, \quad (63)$$

because T^{-1} preserves the order. Hence,

$$T^{-1}\left(\bigvee_{i \in I} T(\hat{P}_i)\right) = \bigvee_{i \in I} \hat{P}_i \quad (64)$$

$$\iff \bigvee_{i \in I} T(\hat{P}_i) = T\left(\bigvee_{i \in I} \hat{P}_i\right), \quad (65)$$

so T preserves all joins. In a completely analogous fashion, one shows that $\bigwedge_{i \in I} T(\hat{P}_i) = T(\bigwedge_{i \in I} \hat{P}_i)$, that is, T also preserves all meets and hence

$$B^{-1}(T) := T|_{\mathcal{P}(\mathcal{N})} \quad (66)$$

is an automorphism of the complete orthomodular lattice $\mathcal{P}(\mathcal{N})$. Interestingly, we only used the fact that T is an involutive map, but not that it is normal.

Conversely, given an automorphism $\tilde{T} : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{N})$, define a partial automorphism $B(\tilde{T}) := T$ from \mathcal{N}_{part} to itself in the obvious way: for self-adjoint operators \hat{A} that are finite real-linear combinations of projections, we have

$$T(\hat{A}) = T\left(\sum_{i=1}^n a_i \hat{P}_i\right) := \sum_{i=1}^n a_i T(\hat{P}_i). \quad (67)$$

An arbitrary self-adjoint \hat{A} can be approximated in norm by a family $(\hat{A}_i)_{i \in \mathbb{N}}$ of self-adjoint operators \hat{A}_i that are finite real-linear combinations of projections, so

$$T(\hat{A}) := \lim_{i \rightarrow \infty} T(\hat{A}_i), \quad (68)$$

where the limit is taken in the norm topology. For a non-self-adjoint normal operator $\hat{B} \in \mathcal{N}_{part}$, use the decomposition $\hat{B} = \hat{A}_1 + i\hat{A}_2$ into self-adjoint operators and let

$$T(\hat{B}) := T(\hat{A}_1) + iT(\hat{A}_2). \quad (69)$$

It remains to show that this T is an automorphism of the partial von Neumann algebra \mathcal{N}_{part} , that is, we need to show that T preserves addition and multiplication of commuting normal operators. (Preservation of the unit element and preservation of the involution are obvious.)

Let \hat{A}, \hat{B} be commuting operators which both are finite real-linear combinations of projections, that is

$$\hat{A} = \sum_{i=1}^n a_i \hat{P}_i, \quad \hat{B} = \sum_{i=1}^n b_i \hat{P}_i. \quad (70)$$

Note that the projections $(\hat{P}_i)_{i=1, \dots, n}$ are pairwise orthogonal, and that \hat{A} and \hat{B} can be written as linear combinations of the *same* projections. We have $\hat{A} + \hat{B} = \sum_{i=1}^n (a_i + b_i) \hat{P}_i$, so

$$T(\hat{A} + \hat{B}) = \sum_{i=1}^n (a_i + b_i) \tilde{T}(\hat{P}_i) \quad (71)$$

$$= \sum_{i=1}^n a_i \tilde{T}(\hat{P}_i) + \sum_{i=1}^n b_i \tilde{T}(\hat{P}_i) \quad (72)$$

$$= T(\hat{A}) + T(\hat{B}). \quad (73)$$

Moreover, $\hat{A}\hat{B} = \sum_{i=1}^n a_i b_i \hat{P}_i$, so

$$T(\hat{A}\hat{B}) = \sum_i a_i b_i \tilde{T}(\hat{P}_i). \quad (74)$$

On the other hand,

$$T(\hat{A})T(\hat{B}) = \sum_i a_i \tilde{T}(\hat{P}_i) \sum_j b_j \tilde{T}(\hat{P}_j) \quad (75)$$

$$= \sum_i a_i (\tilde{T}(\hat{P}_i) \sum_j b_j \tilde{T}(\hat{P}_j)) \quad (76)$$

$$= \sum_i a_i \delta_{ij} b_j \tilde{T}(\hat{P}_j) \quad (77)$$

$$= \sum_i a_i b_i \tilde{T}(\hat{P}_i), \quad (78)$$

so $T(\hat{A}\hat{B}) = T(\hat{A})T(\hat{B})$. By continuity, T can be extended to act as a partial automorphism on all self-adjoint operators in \mathcal{N}_{part} , and by linearity to all normal operators. Hence, $B(\tilde{T}) := T$ is an automorphism of the partial von Neumann algebra \mathcal{N}_{part} .

By construction, $B(\tilde{T})|_{\mathcal{P}(\mathcal{N})} = T|_{\mathcal{P}(\mathcal{N})} = \tilde{T}$, and the two maps B and B^{-1} are inverse to each other, so the groups $\text{Aut}_{cOML}(\mathcal{P}(\mathcal{N}))$ and $\text{Aut}_{part}(\mathcal{N}_{part})$ are isomorphic. ■

If \mathcal{N} is a von Neumann algebra without type I_2 -summand, then by a theorem by Dye [27], there is a group isomorphism

$$C : \text{Aut}_{cOML}(\mathcal{P}(\mathcal{N})) \longrightarrow \text{Aut}_{Jordan}(\mathcal{N}) \quad (79)$$

between the group of automorphisms of the complete orthomodular lattice of projections in \mathcal{N} and the group of Jordan $*$ -automorphisms of \mathcal{N} , seen as a Jordan algebra, that is, as the algebra that has the same elements and linear structure as \mathcal{N} , and Jordan multiplication given by

$$\forall \hat{A}, \hat{B} \in \mathcal{N} : \hat{A} \cdot \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}). \quad (80)$$

A unital Jordan $*$ -morphism preserves the unit element, involution, linear structure and Jordan product. Jordan $*$ -automorphisms are necessarily unital.

Dye's result shows that if $\tilde{T} \in \text{Aut}_{cOML}(\mathcal{P}(\mathcal{N}))$ and $C(\tilde{T}) : \mathcal{N} \rightarrow \mathcal{N}$ is the associated Jordan $*$ -automorphism, then $C(\tilde{T})|_{\mathcal{P}(\mathcal{N})} = \tilde{T}$ (that is, $C(\tilde{T})$ is an extension of \tilde{T}). Note that the partial von Neumann algebra \mathcal{N}_{part} can be seen as 'part' of the the Jordan algebra \mathcal{N} , and that by Prop. 17, we get

$$C(\tilde{T})|_{\mathcal{N}_{part}} = B(\tilde{T}). \quad (81)$$

For this, note that on commuting operators, the Jordan product coincides with the product coming from \mathcal{N} ,

$$\hat{A}\hat{B} = \hat{B}\hat{A} \iff \hat{A} \cdot \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}) = \hat{A}\hat{B}. \quad (82)$$

Moreover, for von Neumann algebras, there is an interesting result connecting twisting maps of the context category $\mathcal{V}(\mathcal{N})$ and Jordan automorphisms of the algebra \mathcal{N} : as was shown in [30],

Theorem 18 *Let \mathcal{N} be a von Neumann algebra with no type I_2 -summand, and let $\mathcal{V}(\mathcal{N})$ be its context category, that is, the set of abelian von Neumann subalgebras of \mathcal{N} , partially ordered under inclusion. There is a group isomorphism*

$$D : \text{Aut}_{\text{ord}}(\mathcal{V}(\mathcal{N})) \longrightarrow \text{Aut}_{\text{Jordan}}(\mathcal{N}) \quad (83)$$

between the group $\text{Aut}_{\text{ord}}(\mathcal{V}(\mathcal{N}))$ of order-automorphisms $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ and the group $\text{Aut}_{\text{Jordan}}(\mathcal{N})$ of Jordan $$ -automorphisms $T : \mathcal{N} \rightarrow \mathcal{N}$.*

The proof proceeds in two main steps: first, using a result by Harding and Navara [31], one shows that every order-automorphism (twisting map) $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ determines a unique automorphism $T : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{N})$ of the projection lattice (and vice versa). Then, by the theorem by Dye already cited above [27], this T extends to a Jordan $*$ -automorphism $T : \mathcal{N} \rightarrow \mathcal{N}$. Conversely, every Jordan $*$ -automorphism $T : \mathcal{N} \rightarrow \mathcal{N}$ determines a unique order-automorphism $\gamma : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ of the context category.

Taken together, the results in this subsection, together with [31] and [30], imply:

Theorem 19 *Let \mathcal{N} be a von Neumann algebra without summand of type I_2 , with projection lattice $\mathcal{P}(\mathcal{N})$, context category $\mathcal{V}(\mathcal{N})$, associated partial von Neumann algebra $\mathcal{N}_{\text{part}}$, associated Jordan algebra also denoted \mathcal{N} , and spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$.*

The four groups $\text{Aut}_{\text{ord}}(\mathcal{V}(\mathcal{N}))$, $\text{Aut}_{\text{cOML}}(\mathcal{P}(\mathcal{N}))$, $\text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}})$ and $\text{Aut}_{\text{Jordan}}(\mathcal{N})$ are isomorphic. Every twisting map $\hat{T} \in \text{Aut}_{\text{ord}}(\mathcal{V}(\mathcal{N}))$ induces a unique automorphism $T : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{N})$ of the complete orthomodular lattice of projections, which extends to a partial von Neumann automorphism $T : \mathcal{N}_{\text{part}} \rightarrow \mathcal{N}_{\text{part}}$ and further to a Jordan $$ -automorphism $T : \mathcal{N} \rightarrow \mathcal{N}$. Conversely, each Jordan $*$ -automorphism restricts to an automorphism of the partial algebra $\mathcal{N}_{\text{part}}$, further to an automorphism of $\mathcal{P}(\mathcal{N})$, and induces an order-automorphism of $\mathcal{V}(\mathcal{N})$.*

Each twisting map $\tilde{T} : \mathcal{V}(\mathcal{N}) \rightarrow \mathcal{V}(\mathcal{N})$ induces an automorphism $\langle \tilde{T}, T^* \rangle : \underline{\Sigma}^{\mathcal{N}} \rightarrow \underline{\Sigma}^{\mathcal{N}}$ of the spectral presheaf, and the group $\text{Aut}(\underline{\Sigma}^{\mathcal{N}})$ of automorphisms of the spectral presheaf is contravariantly isomorphic to the groups $\text{Aut}_{\text{ord}}(\mathcal{V}(\mathcal{N}))$, $\text{Aut}_{\text{cOML}}(\mathcal{P}(\mathcal{N}))$, $\text{Aut}_{\text{part}}(\mathcal{N}_{\text{part}})$ and $\text{Aut}_{\text{Jordan}}(\mathcal{N})$.

This shows that the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$ of a von Neumann algebra \mathcal{N} without summand of type I_2 is ‘rigid’ in the sense that it has exactly as many automorphisms as the underlying base category $\mathcal{V}(\mathcal{N})$. On the algebraic side, these automorphisms correspond to the Jordan $*$ -automorphisms of the algebra \mathcal{N} . Hence, we can consider $\underline{\Sigma}^{\mathcal{N}}$ as a kind of spectrum of the Jordan algebra \mathcal{N} . Note that \mathcal{N} is nonassociative if and only if the underlying von Neumann algebra is nonabelian.³

In terms of reconstruction, we see that the spectral presheaf $\underline{\Sigma}^{\mathcal{N}}$ of a von Neumann algebra contains exactly enough information to determine \mathcal{N} as a Jordan algebra. In terms of a full reconstruction, what is missing is the Lie algebra structure on the self-adjoint elements of \mathcal{N} . In future work, we will consider the question of how to incorporate the Lie algebra structure ‘geometrically’ in the spectral presheaf. This will involve the theory of orientations on operator algebras and their state spaces pioneered by Connes [11] and developed further by Alfsen, Hanche-Olsen, Iochum and Shultz, see [1, 2, 3] and references therein.

5.2 Unital C^* -algebras

Definition 20 Let \mathcal{A} be a unital C^* -algebra, and let $\underline{\Sigma}^{\mathcal{A}}$ be its spectral presheaf. An automorphism of $\underline{\Sigma}^{\mathcal{A}}$ is a pair $\langle \Gamma, \eta \rangle$, where $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ is an essential geometric automorphism, induced by an order-automorphism $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ (called the twisting map). $\Gamma^* : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ is the inverse image functor of the geometric automorphism Γ , and $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \rightarrow \underline{\Sigma}^{\mathcal{A}}$ is a natural isomorphism for which each component $\eta_C : (\Gamma^*(\underline{\Sigma}^{\mathcal{A}}))_C \rightarrow \underline{\Sigma}_C^{\mathcal{A}}$, $C \in \mathcal{C}(\mathcal{A})$, is a homeomorphism. Hence, an automorphism $\langle \Gamma, \eta \rangle$ acts by

$$\underline{\Sigma}^{\mathcal{A}} \xrightarrow{\Gamma^*} \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \xrightarrow{\eta} \underline{\Sigma}^{\mathcal{A}}. \quad (84)$$

We will also use the notation $\eta \circ \Gamma^*$ for an automorphism $\langle \Gamma, \eta \rangle$. The automorphisms of $\underline{\Sigma}^{\mathcal{A}}$ form a group, which we denote as $\text{Aut}(\underline{\Sigma}^{\mathcal{A}})$.

As in the case of von Neumann algebras, instead of requiring that the twisting map $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ is an order-automorphism, we could require

³Moreover, \mathcal{N} is not just a Jordan algebra, but a *JBW*-algebra, that is, an ultraweakly closed Jordan-Banach algebra.

equivalently that γ is an invertible covariant functor, or just require that there is an essential geometric automorphism $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ (which implies that there is an underlying invertible covariant functor $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$).

We will show in Prop. 26 that every automorphism of a unital C^* -algebra \mathcal{A} gives an automorphism of $\underline{\Sigma}^{\mathcal{A}}$ of its spectral presheaf in the sense defined above.

Using a result by Hamhalter [29], we will show that if \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$, then every twisting map γ induces a unique natural isomorphism η and hence a unique automorphism $\langle \Gamma, \eta \rangle : \underline{\Sigma}^{\mathcal{A}} \rightarrow \underline{\Sigma}^{\mathcal{A}}$ of the spectral presheaf, see Thm. 30.

We now characterise automorphisms of $\underline{\Sigma}^{\mathcal{A}}$:

Lemma 21 *Let \mathcal{A} be a unital C^* -algebra, with spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$. Each automorphism $\langle \Gamma, \eta \rangle$ of $\underline{\Sigma}^{\mathcal{A}}$ induces an automorphism $T : \mathcal{A}_{\text{part}} \rightarrow \mathcal{A}_{\text{part}}$ of the partial unital C^* -algebra $\mathcal{A}_{\text{part}}$ of normal operators in \mathcal{A} .*

Proof. Let $\overline{\mathcal{A}}$ be the copresheaf over $\mathcal{C}(\mathcal{A})$ defined in section 4. The twisting map $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ induces an essential geometric automorphism $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ on the topos of copresheaves over $\mathcal{C}(\mathcal{A})$, whose inverse image part Γ^* acts on $\overline{\mathcal{A}}$ by

$$\forall C \in \mathcal{C}(\mathcal{A}) : (\Gamma^*(\overline{\mathcal{A}}))_C = \overline{\mathcal{A}}_{\gamma(C)} = \gamma(C). \quad (85)$$

For each $C \in \mathcal{C}(\mathcal{A})$, the component $\eta_C : (\Gamma^*(\underline{\Sigma}^{\mathcal{A}}))_C \rightarrow \underline{\Sigma}_C^{\mathcal{A}}$ of the natural isomorphism η gives, by Gelfand duality, an isomorphism

$$\begin{aligned} t_C : C &\longrightarrow \gamma(C) \\ \hat{A} &\longmapsto \hat{A} \circ \eta_C \end{aligned} \quad (86)$$

between the abelian C^* -algebras C and $\gamma(C)$. Naturality of η easily implies that $t = (t_C)_{C \in \mathcal{C}(\mathcal{A})}$ is a natural transformation from $\overline{\mathcal{A}}$ to $\Gamma^*(\overline{\mathcal{A}})$, and the fact that η has an inverse $\eta^{-1} : \underline{\Sigma}^{\mathcal{A}} \rightarrow \Gamma^*(\underline{\Sigma}^{\mathcal{A}})$ implies that t has an inverse $t^{-1} : \Gamma^*(\overline{\mathcal{A}}) \rightarrow \overline{\mathcal{A}}$, too. Hence, $t : \overline{\mathcal{A}} \rightarrow \Gamma^*(\overline{\mathcal{A}})$ is a natural isomorphism in the topos $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ for which each component t_C , $C \in \mathcal{C}(\mathcal{A})$, is an isomorphism of abelian C^* -algebras. Define a map

$$\begin{aligned} T : \mathcal{A}_{\text{part}} &\longrightarrow \mathcal{A}_{\text{part}} \\ \hat{A} &\longmapsto t_C(\hat{A}) \end{aligned} \quad (87)$$

from the set of normal operators in \mathcal{A} to itself, where $C \in \mathcal{C}(\mathcal{A})$ is an abelian subalgebra that contains \hat{A} . This map is well-defined (that is, the value $T(\hat{A}) = t_C(\hat{A})$ does not depend on which algebra C containing \hat{A} we choose), since $t = (t_C)_{C \in \mathcal{C}(\mathcal{A})}$ is natural, and the copresheaf maps of $\overline{\mathcal{A}}$ are simply inclusions (see Def. 5). Clearly, $T(\hat{1}) = \hat{1}$.

For all normal operators \hat{A} , we have $T(\hat{A}^*) = t_C(\hat{A}^*) = t_C(\hat{A})^* = T(\hat{A})^*$, where $C \in \mathcal{C}(\mathcal{A})$ such that $\hat{A} \in C$. If \hat{A}, \hat{B} are commuting normal operators, then there exists some $C \in \mathcal{C}(\mathcal{A})$ containing them both, and

$$T(\hat{A} + \hat{B}) = t_C(\hat{A} + \hat{B}) = t_C(\hat{A}) + t_C(\hat{B}) = T(\hat{A}) + T(\hat{B}). \quad (88)$$

Analogously, $T(\hat{A}\hat{B}) = T(\hat{A})T(\hat{B})$.

It is clear by construction that $T(C) = t_C(C) = \gamma(C)$ for all $C \in \mathcal{C}(\mathcal{A})$.

■

Conversely, we have

Lemma 22 *Every automorphism $T : \mathcal{A}_{part} \rightarrow \mathcal{A}_{part}$ of the unital partial C^* -algebra \mathcal{A}_{part} induces an automorphism $\langle \Gamma, \eta \rangle$ of $\underline{\Sigma}^{\mathcal{A}}$, that is, an order-automorphism $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ with corresponding essential geometric automorphism $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{op}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{op}}$ and a natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \rightarrow \underline{\Sigma}^{\mathcal{A}}$ such that each component η_C is a homeomorphism.*

Proof. Let $C \in \mathcal{C}(\mathcal{A})$ be a unital abelian C^* -subalgebra of \mathcal{A} . Then $T|_C : C \rightarrow \mathcal{A}_{part}$ is a unital $*$ -homomorphism from the abelian C^* -algebra C into \mathcal{A}_{part} , so $T|_C(C)$ is norm-closed and hence a unital abelian C^* -subalgebra of $\mathcal{A}_{part} \subset \mathcal{A}$, that is, $T|_C(C) \in \mathcal{C}(\mathcal{A})$.

Clearly, $C' \subset C$ implies $T|_{C'}(C') \subset T|_C(C)$, so we have a monotone map

$$\begin{aligned} \gamma : \mathcal{C}(\mathcal{A}) &\longrightarrow \mathcal{C}(\mathcal{A}) \\ C &\longmapsto T|_C(C). \end{aligned} \quad (89)$$

Moreover, since T has an inverse T^{-1} that is a unital partial $*$ -automorphism as well, the map γ has an inverse $\gamma^{-1} : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$, too, and hence γ is an order-automorphism (twisting map).

Since T is an automorphism, $T|_C : C \rightarrow T(C)$ is isomorphism of unital abelian C^* -algebras. There is a natural isomorphism $t : \overline{\mathcal{A}} \rightarrow \Gamma^*(\overline{\mathcal{A}})$ in $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$, with components

$$\forall C \in \mathcal{C}(\mathcal{A}) : t_C := T|_C : C \longrightarrow T(C). \quad (90)$$

Define, for each $C \in \mathcal{C}(\mathcal{A})$,

$$\begin{aligned} \eta_C : (\Gamma^*(\underline{\Sigma}^{\mathcal{A}}))_C = \underline{\Sigma}_{\gamma(C)}^{\mathcal{A}} &\longrightarrow \underline{\Sigma}_C^{\mathcal{A}} \\ \lambda &\longmapsto \lambda \circ T|_C \end{aligned} \quad (91)$$

the homeomorphism between the Gelfand spectra of $\gamma(C)$ and C corresponding to $T|_C$. By construction, the η_C are the components of a natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \rightarrow \underline{\Sigma}^{\mathcal{A}}$. ■

Clearly, the constructions in Lemma 21 and Lemma 22 are inverse to each other. If $\langle \Gamma_1, \eta_1 \rangle, \langle \Gamma_2, \eta_2 \rangle$ are two automorphisms of $\underline{\Sigma}^{\mathcal{A}}$, and T_1, T_2 are the corresponding partial $*$ -automorphisms of \mathcal{A}_{part} , then by construction, the partial $*$ -automorphism corresponding to the composite automorphism $\langle \Gamma_1, \eta_1 \rangle \circ \langle \Gamma_2, \eta_2 \rangle$ is $T_2 \circ T_1$. Summing up, we have shown:

Proposition 23 *There is a contravariant group isomorphism between $\text{Aut}(\underline{\Sigma}^{\mathcal{A}})$, the group of automorphisms of $\underline{\Sigma}^{\mathcal{A}}$, and $\text{Aut}_{part}(\mathcal{A}_{part})$, the group of unital partial $*$ -automorphisms of \mathcal{A}_{part} .*

Definition 24 *Let \mathcal{A} be a unital C^* -algebra, and let $\overline{\mathcal{A}}$ be its Bohrification (see Def. 5). An automorphism of $\overline{\mathcal{A}}$ is a pair $\langle \Gamma, \kappa \rangle$, where $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ is an essential geometric automorphism, induced by an order-automorphism $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ (the twisting map). $\Gamma^* : \mathbf{Set}^{\mathcal{C}(\mathcal{A})} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ is the inverse image functor of the geometric automorphism Γ , and $\kappa : \overline{\mathcal{A}} \rightarrow \Gamma^*(\overline{\mathcal{A}})$ is a natural isomorphism for which each component $\kappa_C : \overline{\mathcal{A}}_C \rightarrow \Gamma^*(\overline{\mathcal{A}})_C$, $C \in \mathcal{C}(\mathcal{A})$, is a unital $*$ -isomorphism (of abelian C^* -algebras). An automorphism $\langle \Gamma, \kappa \rangle$ acts by*

$$\overline{\mathcal{A}} \xrightarrow{\Gamma^*} \Gamma^*(\overline{\mathcal{A}}) \xrightarrow{\kappa^{-1}} \overline{\mathcal{A}}. \quad (92)$$

The automorphisms of $\overline{\mathcal{A}}$ form a group, which we denote as $\text{Aut}(\overline{\mathcal{A}})$.

An automorphism of \mathcal{A} as an object of $\mathbf{Set}^{\mathcal{C}(\mathcal{A})}$ in the sense defined above corresponds to an automorphism of $\overline{\mathcal{A}}$ as an object of $\mathbf{ucC}^{*\mathcal{C}(\mathcal{A})}$, and also to an automorphism of $\overline{\mathcal{A}}$ as an object of $\mathbf{Coprosh}(\mathbf{ucC}^*)$. Prop. 10 implies:

Corollary 25 *There is a contravariant group isomorphism*

$$\text{Aut}(\underline{\Sigma}^{\mathcal{A}}) \longrightarrow \text{Aut}(\overline{\mathcal{A}}) \quad (93)$$

Noting that \mathcal{A}_{part} is the ‘external’ description of the topos-internal abelian algebra $\overline{\mathcal{A}} \in \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{op}}$ (cf. Rem. 7), one can read the proofs of Lemmas 21 and 22 as making this correspondence between automorphisms of $\underline{\Sigma}^{\mathcal{A}}$ and automorphisms of $\overline{\mathcal{A}}$ explicit.

The following result links automorphisms of a unital C^* -algebra \mathcal{A} and automorphisms of its spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$:

Proposition 26 *Let \mathcal{A} be a unital C^* -algebra, and let $\underline{\Sigma}^{\mathcal{A}}$ be its spectral presheaf. There is an injective group homomorphism*

$$\begin{aligned} \mathrm{Aut}(\mathcal{A}) &\longrightarrow \mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})^{op} \\ \phi &\longmapsto \langle \Phi, \mathcal{G}_\phi \rangle = \mathcal{G}_\phi \circ \Phi^* \end{aligned} \tag{94}$$

from the automorphism group of \mathcal{A} into the opposite group of automorphism group of $\underline{\Sigma}^{\mathcal{A}}$. (This is the same as an injective, contravariant group homomorphism from $\mathrm{Aut}(\mathcal{N})$ into $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{N}})$.)

Proof. The result is a corollary of what we proved so far. It follows from Prop. 10 and Thm. 11 (though injectivity would need a separate argument). More explicitly, one can argue: let $\phi : \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism. By restricting ϕ to the partial C^* -algebra \mathcal{A}_{part} , we obtain an automorphism $\phi|_{\mathcal{A}_{part}}$ of \mathcal{A}_{part} , and by Prop. 23, this corresponds to an automorphism of $\underline{\Sigma}^{\mathcal{A}}$. By the fact that the groups $\mathrm{Aut}_{part}(\mathcal{A}_{part})$ and $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})$ are contravariantly isomorphic, this gives an injective, contravariant group homomorphism from $\mathrm{Aut}(\mathcal{A})$ into $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})$ (which is the same as a group homomorphism from $\mathrm{Aut}(\mathcal{N})$ into $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{N}})^{op}$).

If $\phi, \xi : \mathcal{A} \rightarrow \mathcal{A}$ are two distinct automorphisms of \mathcal{A} , then there exists some self-adjoint operator $\hat{A} \in \mathcal{A}_{part}$ for which $\phi(\hat{A}) \neq \xi(\hat{A})$, so the two automorphisms $\phi|_{\mathcal{A}_{part}}$ and $\xi|_{\mathcal{A}_{part}}$ of the partial C^* -algebra \mathcal{A}_{part} are distinct, too, and the group homomorphism from $\mathrm{Aut}(\mathcal{A})$ into $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})^{op}$ is injective. ■

Yet, $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})$ has more elements than those corresponding to elements of $\mathrm{Aut}(\mathcal{A})$. Each $\phi \in \mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})$ induces a unital partial $*$ -automorphism $p \in \mathrm{Aut}_{part}(\mathcal{A}_{part})$ of the partial C^* -algebra \mathcal{A}_{part} of normal elements, but not every unital partial $*$ -automorphism p induces a C^* -automorphism $\phi \in \mathrm{Aut}(\mathcal{A})$.

Remark 27 *If we aim to reconstruct the algebra \mathcal{A} from its spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ (or, more realistically, from some related presheaf), a key task is to understand the automorphism group $\mathrm{Aut}(\underline{\Sigma}^{\mathcal{A}})$ in detail. We have shown that*

the map

$$\begin{aligned} \text{Aut}(\mathcal{A}) &\longrightarrow \text{Aut}(\underline{\Sigma}^{\mathcal{A}})^{\text{op}} \\ \phi &\longmapsto \langle \Phi, \mathcal{G}_\phi \rangle = \mathcal{G}_\phi \circ \Phi^* \end{aligned} \quad (95)$$

is injective, but not surjective. In this perspective, $\underline{\Sigma}^{\mathcal{A}}$ has too many automorphisms, which means that it has not enough structure to serve as a proper ‘dual space’ of the nonabelian algebra \mathcal{A} that would allow a full reconstruction.

In the arguments above, we started from an automorphism $\langle \Gamma, \eta \rangle$ of $\underline{\Sigma}^{\mathcal{A}}$, where the geometric automorphism $\Gamma : \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{C}(\mathcal{A})^{\text{op}}}$ is induced by a twisting map $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ and the natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \rightarrow \underline{\Sigma}^{\mathcal{A}}$ is given as additional data.

It is also interesting to start just from an order-automorphism $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$, without assuming that a natural isomorphism $\eta : \Gamma^*(\underline{\Sigma}^{\mathcal{A}}) \rightarrow \underline{\Sigma}^{\mathcal{A}}$ is given. One may ask how much operator-algebraic structure can be reconstructed from the poset $\mathcal{C}(\mathcal{A})$ of unital abelian C^* -subalgebras of a nonabelian C^* -subalgebra \mathcal{A} . (For the case of von Neumann algebras, see Thm. 18.)

There is an interesting recent result by Jan Hamhalter [29] on reconstructing parts of the algebraic structure of a unital C^* -algebra \mathcal{A} from its poset $\mathcal{C}(\mathcal{A})$ of unital abelian subalgebras. We need a definition first: let \mathcal{A}, \mathcal{B} be unital C^* -algebras, and let \mathcal{A}_{sa} be the unital Jordan algebra of self-adjoint elements in \mathcal{A} , with Jordan product given by

$$\forall \hat{A}, \hat{B} \in \mathcal{A}_{\text{sa}} : \hat{A} \cdot \hat{B} := \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}). \quad (96)$$

The unital Jordan algebra \mathcal{B}_{sa} is defined analogously. Hamhalter defines a *quasi-Jordan homomorphism* to be a unital map

$$Q : \mathcal{A}_{\text{sa}} \longrightarrow \mathcal{B}_{\text{sa}} \quad (97)$$

such that, for all $C \in \mathcal{C}(\mathcal{A})$,

$$Q|_{C_{\text{sa}}} : C_{\text{sa}} \longrightarrow \mathcal{B}_{\text{sa}} \quad (98)$$

is a unital Jordan homomorphism. Note that Q is only required to be linear on commuting self-adjoint operators, so it is a quasi-linear map. Moreover, Q preserves the Jordan product on commuting operators (where the Jordan product coincides with the operator product since

$$\hat{A}\hat{B} = \hat{B}\hat{A} \iff \hat{A} \cdot \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}) = \hat{A}\hat{B}, \quad (99)$$

as we had already observed for the case of von Neumann algebras).

A *quasi-Jordan isomorphism* is a bijective map $Q : \mathcal{A}_{\text{sa}} \rightarrow \mathcal{B}_{\text{sa}}$ such that Q and Q^{-1} are quasi-Jordan homomorphisms.

Theorem 28 (Hamhalter [29]) *Let \mathcal{A}, \mathcal{B} be unital C^* -algebras such that \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$. There is an order-isomorphism*

$$\gamma : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{B}) \quad (100)$$

if and only if there is a unital quasi-Jordan isomorphism

$$Q : \mathcal{A}_{\text{sa}} \longrightarrow \mathcal{B}_{\text{sa}}. \quad (101)$$

The unital quasi-Jordan algebra \mathcal{A}_{sa} is closely related to our unital partial C^* -algebra $\mathcal{A}_{\text{part}}$, it is simply the self-adjoint part of it. Moreover, each automorphism T of $\mathcal{A}_{\text{part}}$ obviously restricts to a unital quasi-Jordan automorphism $T|_{\mathcal{A}_{\text{sa}}}$ of \mathcal{A}_{sa} . Conversely, every unital quasi-Jordan automorphism Q of \mathcal{A}_{sa} extends to an automorphism of $\mathcal{A}_{\text{part}}$ by linearity. This means there is a group isomorphism

$$\text{Aut}_{\text{quJord}}(\mathcal{A}_{\text{sa}}) \longrightarrow \text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}}), \quad (102)$$

where $\text{Aut}_{\text{quJord}}(\mathcal{A}_{\text{sa}})$ is the group of quasi-Jordan automorphisms of \mathcal{A}_{sa} . Using this, Thm. 28 can be reformulated as

Theorem 29 *Let \mathcal{A} be a unital C^* -algebra such that \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$. There are group isomorphisms*

$$\text{Aut}_{\text{ord}}(\mathcal{C}(\mathcal{A})) \longrightarrow \text{Aut}_{\text{quJord}}(\mathcal{A}_{\text{sa}}) \longrightarrow \text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}}). \quad (103)$$

This shows that if \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$, then every twisting map $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ induces a unique automorphism of the unital partial C^* -algebra $\mathcal{A}_{\text{part}}$. From Prop. 23 and Cor. 25, we obtain

Theorem 30 *Let \mathcal{A} be a unital C^* -algebra such that \mathcal{A} is neither isomorphic to \mathbb{C}^2 nor to $\mathcal{B}(\mathbb{C}^2)$. The four groups $\text{Aut}_{\text{ord}}(\mathcal{C}(\mathcal{A}))$, $\text{Aut}_{\text{quJord}}(\mathcal{A}_{\text{sa}})$, $\text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}})$ and $\text{Aut} \overline{\mathcal{A}}$ are isomorphic, and these groups are contravariantly isomorphic to the group $\text{Aut}(\underline{\Sigma}^{\mathcal{A}})$ of automorphisms of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of \mathcal{A} .*

Note that this result is weaker than the one for von Neumann algebras (Thm. 19), since automorphisms of the spectral presheaf only determine the original algebra as a quasi-Jordan algebra up to isomorphisms, or equivalently, as a unital partial C^* -algebra \mathcal{A}_{part} up to isomorphisms.

But, crucially, also for a unital C^* -algebra \mathcal{A} , every twisting map $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$ induces a unique automorphism of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of \mathcal{A} , so the spectral presheaf is ‘rigid’ in the sense that it has exactly as many automorphisms as the underlying base category $\mathcal{C}(\mathcal{A})$.

For a certain class of unital C^* -algebras (which is strictly larger than the class of von Neumann algebras without type I_2 -summand), a stronger result can be obtained and the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ can be shown to determine the algebra \mathcal{A} up to Jordan isomorphisms.

We note that a *linear* unital quasi-Jordan automorphism $T : \mathcal{A}_{sa} \rightarrow \mathcal{A}_{sa}$ is in fact a Jordan automorphism, that is, it preserves the Jordan product also on noncommuting operators $\hat{A}, \hat{B} \in \mathcal{A}_{sa}$.

To see this,⁴ first note that for all $\hat{A} \in \mathcal{A}_{sa}$, it holds that $T(\hat{A}^2) = T(\hat{A})^2$, since each self-adjoint operator \hat{A} is contained in some $C \in \mathcal{C}(\mathcal{A})$. Moreover, for all $\hat{A}, \hat{B} \in \mathcal{A}_{sa}$, the Jordan product can be written as

$$\hat{A} \cdot \hat{B} = \frac{1}{2}((\hat{A} + \hat{B})^2 - \hat{A}^2 - \hat{B}^2). \quad (104)$$

Then, using linearity of T ,

$$T(\hat{A} \cdot \hat{B}) = T\left(\frac{1}{2}((\hat{A} + \hat{B})^2 - \hat{A}^2 - \hat{B}^2)\right) \quad (105)$$

$$= \frac{1}{2}(T(\hat{A} + \hat{B})^2 - T(\hat{A}^2) - T(\hat{B}^2)) \quad (106)$$

$$= \frac{1}{2}((T(\hat{A}) + T(\hat{B}))^2 - T(\hat{A})^2 - T(\hat{B})^2) \quad (107)$$

$$= T(\hat{A}) \cdot T(\hat{B}). \quad (108)$$

Following [29], the question hence is: which unital quasi-linear Jordan automorphisms $\mathcal{A}_{sa} \rightarrow \mathcal{A}_{sa}$ are linear? The problem which quasi-linear maps from an operator algebra into the complex numbers (or, more generally, into a Banach space) are linear has attracted substantial efforts by many authors and eventually led to the following deep result by Bunce and Wright:

⁴I thank Jan Hamhalter for pointing out this argument to me. – We note in passing that this implies that every partial unital C^* -algebra (respectively partial von Neumann algebra) for which addition is defined globally and not just between commuting elements is a unital Jordan algebra, and more precisely a *JB*-algebra (respectively *JBW*-algebra).

Theorem 31 (*Generalised Gleason Theorem, Bunce and Wright [10]*) *Let \mathcal{N} be a von Neumann algebra with no type I_2 -summand, and let X be a Banach space. Let \mathcal{A} be a C^* -algebra that is a quotient of a norm-closed two-sided ideal I in \mathcal{N} . Suppose that $T : \mathcal{N}_{\text{sa}} \rightarrow X$ is a quasi-linear map that is bounded on the unit ball. Then T is linear.*

As a corollary, Hamhalter obtains

Corollary 32 (*Cor. 3.6, [29]*) *Let \mathcal{N} be a von Neumann algebra with no type I_2 -summand, and let \mathcal{A} be a C^* -algebra that is an at least three-dimensional quotient of an ideal algebra $\hat{1} + I$, where I is a norm-closed two-sided ideal in \mathcal{N} . Let \mathcal{B} be a C^* -algebra. For each order-isomorphism $\gamma : \mathcal{C}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{B})$, there is a unique Jordan isomorphism $T : \mathcal{A}_{\text{sa}} \rightarrow \mathcal{B}_{\text{sa}}$ that induces γ .*

We see that for the class of C^* -algebras described in Cor. 32 (which clearly contains all von Neumann algebras with no type I_2 -summand), the group $\text{Aut}_{\text{quJord}}(\mathcal{A}_{\text{sa}})$ of unital quasi-Jordan automorphisms in fact is the group $\text{Aut}_{\text{Jordan}}(\mathcal{A}_{\text{sa}})$. Each Jordan automorphism $T \in \text{Aut}_{\text{Jordan}}(\mathcal{A}_{\text{sa}})$ can be extended by linearity to a Jordan $*$ -automorphism of \mathcal{A} . Conversely, each Jordan $*$ -automorphism of \mathcal{A} restricts to a Jordan automorphism of \mathcal{A}_{sa} , and the two maps are inverse to each other. So, the groups $\text{Aut}_{\text{Jordan}}(\mathcal{A}_{\text{sa}})$ and $\text{Aut}_{\text{Jordan}}(\mathcal{A})$ are isomorphic.

Hence, for the particular class of unital C^* -algebras described in Cor. 32, the spectral presheaf determines the Jordan structure completely, and we obtain

Corollary 33 *Let \mathcal{N} be a von Neumann algebra with no type I_2 -summand, and let \mathcal{A} be a C^* -algebra that is an at least three-dimensional quotient of an ideal algebra $\hat{1} + I$, where I is a norm-closed two-sided ideal in \mathcal{N} . The groups $\text{Aut}_{\text{ord}}(\mathcal{C}(\mathcal{A}))$, $\text{Aut}_{\text{Jordan}}(\mathcal{A}_{\text{sa}})$, $\text{Aut}_{\text{part}}(\mathcal{A}_{\text{part}})$, $\text{Aut}_{\text{Jordan}}(\mathcal{A})$ and $\text{Aut } \overline{\mathcal{A}}$ are isomorphic, and these groups are contravariantly isomorphic to the group $\text{Aut}(\underline{\Sigma}^{\mathcal{A}})$ of automorphisms of the spectral presheaf $\underline{\Sigma}^{\mathcal{A}}$ of \mathcal{A} .*

Acknowledgements. Discussions with Chris Isham, Boris Zilber and Yuri Manin are gratefully acknowledged. Rui Soares Barbosa provided key input, and Tom Woodhouse, Nadish de Silva, Dan Marsden and Carmen Constantin gave valuable feedback, for which I thank them. I also thank Bertfried Fauser, who carefully read a draft and made a number of very helpful remarks. Jan Hamhalter provided inspiration and help with a technical point, for which I am grateful. I also thank John Harding, Izumi Ojima and Dirk Pattinson for their interest in this work (and for their patience).

References

- [1] E.M. Alfsen, F.W. Shultz, “Orientation in operator algebras”, *Proc. Natl. Acad. Sci. USA* **95**, 6596–6601 (1998).
- [2] E.M. Alfsen, F.W. Shultz, *State Spaces of Operator Algebras, Basic Theory, Orientations and C^* -products*, Birkhäuser, Boston (2001).
- [3] E.M. Alfsen, F.W. Shultz, *Geometry of State Spaces of Operator Algebras*, Birkhäuser, Boston (2003).
- [4] B. Banaschewski, C.J. Mulvey, “A constructive proof of the Stone-Weierstrass theorem”, *J. Pure Appl. Algebra* **116**, 25–40 (1997).
- [5] B. Banaschewski, C.J. Mulvey, “The spectral theory of commutative C^* -algebras: the constructive spectrum”, *Quaest. Math.* **23**, 425–464 (2000).
- [6] B. Banaschewski, C.J. Mulvey, “The spectral theory of commutative C^* -algebras: the constructive Gelfand-Mazur theorem”, *Quaest. Math.* **23**, 465–488 (2000).
- [7] B. Banaschewski, C.J. Mulvey, “A globalisation of the Gelfand duality theorem.”, *Ann. Pure & Applied Logic* **137**, 62–103 (2006).
- [8] B. van den Berg, C. Heunen, “Noncommutativity as a colimit”, arXiv:1003.3618v3 (2010; version 3 from 25. April 2012).
- [9] B. Blackadar, *Operator Algebras, Theory of C^* -Algebras and von Neumann Algebras*, Springer, Berlin, Heidelberg (2006).
- [10] L.J. Bunce, J.D. Maitland Wright, “Velocity maps in von Neumann algebras”, *Pacific J. Math.* **170**, No. 2, 421–427 (1995).
- [11] A. Connes, “Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann”, *Annales de l’institut Fourier* **24**, No. 4, 121–155 (1974).
- [12] A. Döring, “Quantum States and Measures on the Spectral Presheaf”, *Adv. Sci. Lett.* **2**, special issue on “Quantum Gravity, Cosmology and Black Holes”, ed. M. Bojowald, 291–301 (2009).
- [13] A. Döring, “Topos theory and ‘neo-realist’ quantum theory”, in *Quantum Field Theory, Competitive Models*, eds. B. Fauser, J. Tolksdorf, E. Zeidler, Birkhäuser, Basel, Boston, Berlin (2009).

- [14] A. Döring, “Topos quantum logic and mixed states”, in *Proceedings of the 6th International Workshop on Quantum Physics and Logic (QPL 2009)*, *Electronic Notes in Theoretical Computer Science* **270**, No. 2 (2011).
- [15] A. Döring, “The Physical Interpretation of Daseinisation”, in *Deep Beauty*, ed. Hans Halvorson, Cambridge University Press, Cambridge, 207–238 (2011).
- [16] A. Döring, “Topos-based Logic for Quantum Systems and Bi-Heyting Algebras”, to appear in *Logic & Algebra in Quantum Computing*, Lecture Notes in Logic, Association for Symbolic Logic in conjunction with Cambridge University Press; arXiv:1202.2750 (2012).
- [17] A. Döring, “Generalised Gelfand Spectra of Nonabelian Unital C^* -Algebras II: Flows and Time Evolution of Quantum Systems”, arXiv (2012).
- [18] A. Döring, R. Soares Barbosa, “Unsharp Values, Domains and Topoi”, in *Quantum Field Theory and Gravity, Conceptual and Mathematical Advances in the Search for a Unified Framework*, eds. F. Finster et al., Birkhäuser, Basel, 65–96 (2012).
- [19] A. Döring, B. Dewitt, “Self-adjoint Operators as Functions I: Lattices, Galois Connections, and the Spectral Order”, arXiv:1208.4724 (2012).
- [20] A. Döring, B. Dewitt, “Self-adjoint Operators as Functions II: Quantum Probability”, arXiv:1210.5747 (2012).
- [21] A. Döring, C.J. Isham, “A topos foundation for theories of physics: I. Formal languages for physics”, *J. Math. Phys.* **49**, Issue 5, 053515 (2008).
- [22] A. Döring, C.J. Isham, “A topos foundation for theories of physics: II. Daseinisation and the liberation of quantum theory”, *J. Math. Phys.* **49**, Issue 5, 053516 (2008).
- [23] A. Döring, C.J. Isham, “A topos foundation for theories of physics: III. Quantum theory and the representation of physical quantities with arrows $\delta(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$ ”, *J. Math. Phys.* **49**, Issue 5, 053517 (2008).
- [24] A. Döring, C.J. Isham, “A topos foundation for theories of physics: IV. Categories of systems”, *J. Math. Phys.* **49**, Issue 5, 053518 (2008).

- [25] A. Döring, and C.J. Isham, “‘What is a Thing?’: Topos Theory in the Foundations of Physics”, in *New Structures for Physics*, ed. B. Coecke, Lecture Notes in Physics **813**, Springer, Heidelberg, Dordrecht, London, New York, 753–937 (2011).
- [26] A. Döring, C.J. Isham, “Classical and Quantum Probabilities as Truth Values”, *J. Math. Phys.* **53**, 032101 (2012).
- [27] H.A. Dye, “On the Geometry of Projections in Certain Operator Algebras”, *Annals of Mathematics* **61**, No. 1, 73–89 (1955).
- [28] J. Funk, private communication (2012).
- [29] J. Hamhalter, “Isomorphisms of ordered structures of abelian C^* -subalgebras of C^* -algebras”, *J. Math. Anal. Appl.* **383**, 391–399 (2011).
- [30] J. Harding, A. Döring, “Abelian subalgebras and the Jordan structure of a von Neumann algebra”, arXiv:1009.4945 (2010).
- [31] J. Harding, M. Navara, “Subalgebras of orthomodular lattices”, *Order* **28**, No. 3, 549–563 (2011).
- [32] C. Heunen, N.P. Landsman, B. Spitters, “A topos for algebraic quantum theory”, *Comm. Math. Phys.* **291**, 63–110 (2009).
- [33] C. Heunen, N.P. Landsman, B. Spitters, “Bohrification of von Neumann algebras and quantum logic”, *Synthese*, online first, DOI: 10.1007/s11229-011-9918-4 (2011).
- [34] C. Heunen, N.P. Landsman, B. Spitters, “Bohrification”, in *Deep Beauty*, ed. H. Halvorson, Cambridge University Press, 271–313 (2011).
- [35] C.J. Isham, J. Butterfield, “A topos perspective on the Kochen-Specker theorem: I. Quantum states as generalised valuations”, *Int. J. Theor. Phys.* **37**, 2669–2733 (1998).
- [36] C.J. Isham, J. Hamilton, J. Butterfield, “A topos perspective on the Kochen-Specker theorem: III. Von Neumann algebras as the base category”, *Int. J. Theor. Phys.* **39**, 1413–1436 (2000).
- [37] P.T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Vols. 1&2, Oxford Logic Guides **43&44**, Oxford University Press, Oxford (2002/03).

- [38] R.V. Kadison, J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Volume 1: Elementary Theory* and *Volume 2: Advanced Theory*, Academic Press, New York (1983/86).
- [39] S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer, New York, Berlin, Heidelberg (1992).
- [40] nLab, “Ringed topos”, <http://ncatlab.org/nlab/show/ringed+topos>, retrieved on May 11, 2012.
- [41] N. de Silva, “Extending geometric invariants from topological spaces to noncommutative spaces”, transfer thesis, University of Oxford (2012).
- [42] S. Wolters, “A Comparison of Two Topos-Theoretic Approaches to Quantum Theory”, arXiv:1010.2031v2 (version 2 from 3. August 2011).
- [43] T. Woodhouse, “Time Evolution in Quantum Theory and Quantum Information, A Topos Theoretic Perspective”, MSc thesis, University of Oxford (2011).